# The Reliability of a Homogeneous Test

Measurement Methods

Lecture 12

# Today's Class

- Estimating the reliability of a homogeneous test.
  - McDonald's Omega.
  - Guttman-Chronbach Alpha.
  - Spearman-Brown.
- What to do with binary items.
  - Wait until we talk about more appropriate methods for binary items.
- Brief notes about generalizability.

# The Reliability of a Homogeneous Test

# The Single-Factor Model for Test Scores

- Recall from the previous chapter our discussion was framed about how to estimate the precision of a test.

- The classical true-score model was introduced as a method to get at reliability and the standard error of measurement.

- We will now see what happens when we combine a total test score with the use of the single-factor model.

# Total Test Score

- Taking what we know about our single-factor model, let's see if we can express the total test score in terms of the model:

- The test score is: $Y = \sum_j X_j$

- The single factor model states: $X_j = \mu_j + \lambda_j F + E_j$

# Putting the Two Together

- So plugging in for X we get:

$$Y = \sum_j X_j = \sum_j (\mu_j + \lambda_j F + E_j)$$
$$= \sum_j \mu_j + \sum_j (\lambda_j) F + \sum_j E_j$$

- We can now re-write into:

$$Y = C + U + \mu_y$$

- C is the common part of Y due to the common factor.
- U is the unique part due to the idiosyncratic properties of the test.
- $\mu_Y$ is the mean of the test Y.

# Doesn't That Look Familiar?

- So, we are now back to looking at a test as a decomposition into a "true score" and an "error."
  - The previous slide looked similar to $Y = T + E$
- The factor model gives us an interpretation of the total true score, $T_Y$ as the common part C of Y, and of the error E as the unique part U.
  - So we can rewrite our model as:

$$Y = T_Y + E_Y + \mu_Y$$

# Where Do We Go From Here?

- You can guess what is coming now, can't you?
  - In Chapter 5, we defined the classical true-score model, then went and showed the variance of Y as a decomposition into true and error variances.

- Under the single factor model, we can also show the variance of Y.
  - Only now, our decomposition will be having parameters of our single-factor model.

# Variance of Y

- Using the algebra of expectations, we have:

$$
\begin{aligned}
Var(Y) &= Var[\mu_Y + \Sigma\,(\lambda_j)\,F + \Sigma(E_j)] \\
&= Var[\Sigma\,(\lambda_j)\,F + \Sigma(E_j)] \\
&= [\Sigma\,(\lambda_j)]^2\,Var[F] + Var[\Sigma(E_j)] + 2Cov[F, \Sigma(E_j)] \\
&= [\Sigma\,(\lambda_j)]^2 + \Sigma\,\Psi_j^{\,2} \\
&= \sigma_{Ty}^{\,2} + \sigma_{Ey}^{\,2}
\end{aligned}
$$

# Reliability Coefficient

- So, we have $\sigma_Y{}^2 = \sigma_{Ty}{}^2 + \sigma_{Ey}{}^2$
- Recall the reliability coefficient defined in Chapter 5 was:

$$\rho_r = \sigma_{Ty}{}^2 / (\sigma_{Ty}{}^2 + \sigma_{Ey}{}^2) = \sigma_{Ty}{}^2 / \sigma_Y{}^2$$

- We now define a new version of the reliability coefficient, constructed with the single-factor model in mind.

# McDonald's Omega

- McDonald's Omega is a measure of reliability of a homogeneous test based on the single-factor model:

$$\omega = \sigma_{Ty}^{2} / \sigma_{Y}^{2} = [\Sigma (\lambda_j)]^2/(\sigma_{Y}^{2})$$

-or-

$$\omega = \sigma_{Ty}^{2} / \sigma_{Y}^{2} = [\Sigma (\lambda_j)]^2/([\Sigma (\lambda_j)]^2 + \Sigma \Psi_j^{2})$$

- Omega is a ratio of the true-score variance of Y to the total variance of Y.

# Estimation of McDonald's Omega

- To get an estimate of McDonald's Omega, we must first fit our single-factor model.
- If the fit is adequate – and hence, we have a homogeneous test – we then take our factor loading parameters:
  - sum them
  - then square the sum
  - then divide by the variance of Y (variance of a total test score).

# Properties of McDonald's Omega

- Omega is the ration of the variance due to the common factor to the total variance of Y.

- Omega is the *square* of the correlation between Y and the common factor F.

# More Properties of McDonald's Omega

- Omega is the correlation between two test scores Y and Y' that have the same sum of their loadings and the same sum of their unique variances.

- Omega is the square of the correlation between the total (or mean) score on the given *m* items and the means score on an infinite set of items from a homogeneous domain of items of which the *m* items in the test are a subset.
  - In this respect, it is a measure of the generaliziablity of the test items.

# Reliability of SWLS Data

- Recall our estimates of the SWLS data parameters.
  - From these we can compute omega.
- Sum of the loadings = 5.69
  - Square of 5.69 = 32.25
- Sum of the unique variances = 7.13
- Omega = 32.25/(32.25+7.13) = 0.82

TABLE 6.4
Satisfaction With Life Scale—Spearman Analysis

| | (a) | | (b) | | | | |
|---|---|---|---|---|---|---|---|
| | $\lambda$ | $\psi^2$ | 1 | 2 | 3 | 4 | 5 |
| 1 | 1.290 | 0.901 | 2.565 | 1.424 | 1.481 | 1.328 | 1.529 |
| 2 | 1.104 | 1.274 | 1.424 | 2.493 | 1.267 | 1.051 | 1.308 |
| 3 | 1.148 | 1.144 | 1.481 | 1.267 | 2.462 | 1.093 | 1.360 |
| 4 | 0.952 | 1.863 | 1.328 | 1.051 | 1.093 | 2.769 | 1.128 |
| 5 | 1.185 | 1.951 | 1.529 | 1.308 | 1.360 | 1.128 | 3.355 |

| | (c) | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| 1 | .0 | .135 | .006 | −.033 | −.104 |
| 2 | .135 | .0 | .015 | −.206 | .004 |
| 3 | .006 | .015 | .0 | .035 | −.048 |
| 4 | −.033 | −.206 | .035 | .0 | .195 |
| 5 | −.104 | .004 | −.048 | .195 | .0 |

# Coefficient Alpha

# Another Measure of Reliability

- A much more frequently used measure of reliability is the Guttman-Chronbach Alpha.
  - Note that this is commonly called Chronbach's Alpha.
    - Rod (correctly) cites the first reporting of this coefficient in 1945 by Guttman.

- The Omega coefficient of reliability was based on the general single-factor model.
  - Alpha is based on the true-score equivalence model.

# How Does This Work?

- Consider the computation of coefficient omega under the true-score equivalence model:

$$\omega = \frac{m^2 \lambda^2}{\sigma_Y^2}$$

- For the true-score equivalence model in the SWLS example, $\lambda = 1.134$.
  - So omega for this case was:
    - $5^2 (1.134)^2 / 39.382 = 0.8170$

- Note that the G-C alpha is slightly lower than our estimate of omega.

# G-C Alpha

- It can be shown (see p. 91), that the classical definition of the G-C alpha is equal to what omega is under the true-score equivalence model.

- The classical definition of the G-C alpha is:

$$\hat{\alpha} = \left(\frac{m}{m-1}\right)\left(1 - \frac{\sum_j s_{jj}}{s_Y^2}\right)$$

# Calculating Alpha for the SWLS

- To do this, we must sum the diagonal elements of the covariance matrix:
  - 2.566 + 2.493 + 2.462 + 2.769 + 3.356 = 13.646
- Then we must sum every element in the covariance matrix to get $s_Y^2$.
  - $s_Y^2 = 39.382$
- Alpha is then (6/(6-1))*(1-(13.646/39.382)) = 0.8170

$$\boldsymbol{S} = \begin{bmatrix} 2.566 & 1.560 & 1.487 & 1.195 & 1.425 \\ 1.560 & 2.493 & 1.283 & 0.845 & 1.313 \\ 1.487 & 1.283 & 2.462 & 1.127 & 1.313 \\ 1.195 & 0.845 & 1.127 & 2.769 & 1.323 \\ 1.425 & 1.313 & 1.313 & 1.323 & 3.356 \end{bmatrix}$$

# Omega and Alpha in Practice

- It is difficult to invent a homogeneous test where Omega differs greatly from Alpha.

- But part of the motivation for introducing Omega is the motivation behind the measure.
  - The factor model should be used to check for homogeneity of a test.
    - If a test is roughly homogeneous, then the sum score can be used.

# Other Meaning of Alpha

- Cronbach showed that Alpha gives the average correlation between all possible splits of the items into halves.
  - The split-half method is a classical method for estimating reliability – it yields Y and Y'.
- Alpha is also referred to as the "internal consistency" reliability of a test.
  - Rod recommends against this.

# The Spearman-Brown Prophecy Formula

# Covering All Our Models

- We showed that Omega is a reliability coefficient for the single factor model.
- When the true-score equivalence model is estimated, Omega is equal to the G-C Alpha.
- What about the parallel items model?

# Notes About Parallel Items

- Recall that Under the parallel-items model:
  - the factor loadings constrained to be equal

$$\lambda_1 = \lambda_2 = \ldots = \lambda_j$$

  - the unique variances are constrained to be equal:

$$\Psi_1^2 = \Psi_2^2 = \ldots = \Psi_j^2$$

- In this case, omega for a test is the same as a multiple for omega for a single item.
  - This is because the covariance for each pair of items is the same across all pairs

# The Spearman-Brown Prophecy Formula

- Specifically, the reliability for a single item is:

$$p_1 = \frac{\sigma^2_{T_x}}{\sigma^2_{T_x} + \sigma^2_{E_x}}$$

- The reliability for the whole test is then:

$$p_m = \frac{m\sigma^2_{T_x}}{m\sigma^2_{T_x} + \Psi^2}$$

- The final formula is the Spearman-Brown Prophecy formula.
  – It was designed to prophecy the increase in reliability by doubling the length of a test.

# S-B and Omega

- We can show that Omega happens to be equal to the S-B reliability under the parallel-items model.
  – The S-B formula reported from time to time in the context of building test with split-half methods.

- Really, there is not a good reason for using the S-B formula over Alpha or Omega (p. 95).
  – It is presented here for historical reasons.

# S-B from the SWLS

- To demonstrate the S-B from the SWLS, recall the estimates of our parameters:
  - $\lambda = 1.287$
  - $\Psi^2 = 1.442$

- The formula for coefficient Omega:

$$[\Sigma (\lambda_j)]^2/([\Sigma (\lambda_j)]^2 + \Sigma \Psi_j^2) =$$

$$m^2 \lambda^2/(m^2 \lambda^2 + \Sigma \Psi_j^2) = 0.8169$$

# Another Example – The Hope Scale

# The Hope Scale

- Supplied by Hal Shorey, the Hope Scale consists of 18 items measuring three attributes.
  - We will focus on the Pathways Attribute, as measured by the items to the right.

1. I'm not good at planning how to get things done.*
2. I can think of many ways to get out of a jam.
3. I have difficulty finding ways to solve problems.*
4. I'm not good at coming up with solutions.*
5. I'm good at coming up with new ways to solve problems.
6. I create alternate plans when blocked.

# Is The Test Homogenous?

```
PROC IMPORT OUT= WORK.hope
            DATAFILE= "C:\Documents and Settings\Jonat
            DBMS=EXCEL2000 REPLACE;
     GETNAMES=YES;
RUN;

*single factor for pathways;
proc calis cov data=work.hope residual method=uls;
lineqs
  h4=lambda4 f1 + e4,
  h5=lambda5 f1 + e5,
  h9=lambda9 f1 + e9,
  h12=lambda12 f1 + e12,
  h13=lambda13 f1 + e13,
  h15=lambda15 f1 + e15;
std
  f1=1,
  e4=psi4,
  e5=psi5,
  e9=psi9,
  e12=psi12,
  e13=psi13,
  e15=psi15;
run;
```

```
                    The SAS System              21:09 Monday,

                  The CALIS Procedure
 Covariance Structure Analysis: Least-Squares Estimation

 Fit Function                                        0.1651
 Goodness of Fit Index (GFI)                         0.9919
 GFI Adjusted for Degrees of Freedom (AGFI)          0.9810
 Root Mean Square Residual (RMR)                     0.0887
 Parsimonious GFI (Mulaik, 1989)                     0.5951
```

# What is Omega?

```
                 The CALIS Procedure
Covariance Structure Analysis: Least-Squares Estimation

      Manifest Variable Equations with Estimates


    H4       =    0.9171*f1       +  1.0000 e4
                         lambda4
    H5       =    0.6234*f1       +  1.0000 e5
                         lambda5
    H9       =    0.9728*f1       +  1.0000 e9
                         lambda9
    H12      =    0.9897*f1       +  1.0000 e12
                         lambda12
    H13      =    0.7723*f1       +  1.0000 e13
                         lambda13
    H15      =    0.7178*f1       +  1.0000 e15
                         lambda15


            Variances of Exogenous Variables

            Variable Parameter      Estimate

            f1                       1.00000
            e4        psi4           2.21130
            e5        psi5           1.45741
            e9        psi9           1.15605
            e12       psi12          0.75474
            e13       psi13          1.26163
            e15       psi15          0.78997
```

# What is Alpha?

| A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|
| _TYPE_ | _NAME_ | H4 | H5 | H9 | H12 | H13 | H15 |
| COV | H4 | 3.052461 | 0.343256 | 0.939549 | 0.861857 | 0.770751 | 0.788631 |
| COV | H5 | 0.343256 | 1.846086 | 0.654012 | 0.65766 | 0.594421 | 0.497503 |
| COV | H9 | 0.939549 | 0.654012 | 2.102341 | 1.0889 | 0.623994 | 0.559452 |
| COV | H12 | 0.861857 | 0.65766 | 1.0889 | 1.734173 | 0.701174 | 0.630569 |
| COV | H13 | 0.770751 | 0.594421 | 0.623994 | 0.701174 | 1.858153 | 0.63627 |
| COV | H15 | 0.788631 | 0.497503 | 0.559452 | 0.630569 | 0.63627 | 1.305166 |

Sum of diagonal elements: 11.898

Sum of matrix: 32.594

# Wrapping Up

- Reliability of a test can be estimated from a test itself.
  - The single-factor model has Omega as its estimate of reliability.
  - Omega equals Alpha under the true-score equivalence Model.
- The S-B formula is Omega under the parallel-items model
  - Because of the strict assumptions of the model, we cannot think this will be a good estimate of the reliability.
- Binary items will be dealt with in Chapter 11.

# Next Time

- Midterm review.
  - Bring your questions.