Diagnostic Modeling:
Psychometric Models

NCME 2014 Training Session

Diagnostic Measurement –
Theory, Methods, and Applications

Session 2

Course Website: http://wp.me/p3nkOf-nu

Development of Psychometric Models

- Over the past several years, numerous DCMs have been developed
  - We will focus on DCMs that use latent variables for attributes

- Each DCM makes assumptions about how mastered attributes combine/interact to produce an item response
  - Compensatory/disjunctive/additive models
  - Non-compensatory/conjunctive/non-additive models

- With so many models, analysts have been unsure which model would best fit their purpose
  - Difficult to imagine all items following same assumptions

General Models for Diagnosis

- Recent developments have produced very general diagnostic models
  - General Diagnostic Model (**GDM**; von Davier, 2005)
  - Loglinear Cognitive Diagnosis Model (**LCDM**; Henson, Templin, & Willse, 2009)
    - Focus of this session

- The general DCMs (GDM; LCDM) provide great flexibility
  - Subsume all other latent variable DCMs
  - Allow for both additive and non-additive relationships between attributes and items
  - Sync with other psychometric models allowing for greater understanding of modeling process

Session Overview

- Background information
  - ANOVA models and the LCDM
- Logits explained
- The LCDM
  - Parameter structure
  - One-item demonstration
- LCDM general form
- Linking the LCDM to other earlier-developed DCMs

Notation Used Throughout Session

- **Attributes**: $a = 1,\ldots, A$
- **Respondents**: $r = 1,\ldots,R$
- **Attribute Profiles**: $\boldsymbol{\alpha}_r = [\alpha_{r1}, \alpha_{r2},\ldots, \alpha_{rA}]$
  - α_{ra} is 0 or 1
- **Latent Classes**: $c = 1,\ldots,C$
  - We have $C = 2^A$ latent classes – one for each possible attribute profile
- **Items**: $i = 1,\ldots,I$
  - Restricted to dichotomous item responses (X_{ri} is 0 or 1)
- **Q-matrix**: Elements q_{ia} for an item i and attribute a
  - q_{ia} is 0 or 1

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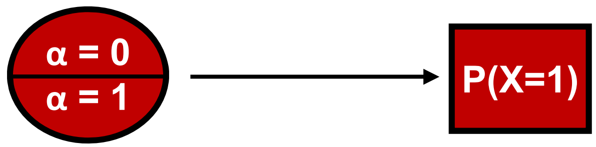
BACKGROUND INFORMATION: ANOVA MODELS

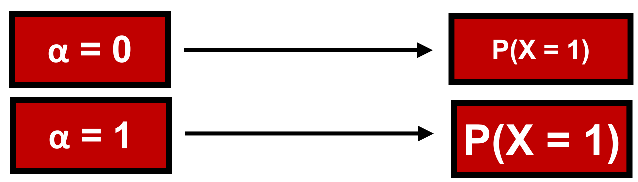
Background Information – ANOVA

- The LCDM models the probability of a correct response to an item as a function of the latent attributes of a respondent



- The latent attributes are categorical, meaning a respondent can have only a few possible statuses
  - Each status corresponds to a predicted probability of a correct response

- As such, the LCDM is very similar to an ANOVA model
  - Predicting the a dependent variable as a function of the experimental group of a respondent



ANOVA Refresher

- As a refresher on ANOVA, lets imagine that we are interested in the factors that have an effect on *work output* (denoted by Y)

- We design a two-factor study where *work output* may be affected by:
  - Lighting of the workplace
    - High or Low
  - Temperature
    - Cold or Warm

- This experimental design is known as a 2-Way ANOVA

ANOVA Model

- Here is the 2 x 2 Factorial design:

	Low Lighting	High Lighting
Cold Temperature	$\bar{Y}_{Cold,Low}$	$\bar{Y}_{Cold,High}$
Warm Temperature	$\bar{Y}_{Warm,Low}$	$\bar{Y}_{Warm,High}$

- The ANOVA model for a respondent's work output is

$$Y_{rtl} = \mu + A_t + B_l + (AB)_{tl} + \varepsilon_{rtl}$$

ANOVA Model

- The ANOVA model allows us to test for the presence of:
  - A main effect associated with *Temperature* (A_t)
  - A main effect associated with *Lighting* (B_l)
  - An interaction effect associated with *Temperature* and *Lighting* $(AB)_{tl}$

$$Y_{rtl} = \mu + A_t + B_l + (AB)_{tl} + \varepsilon_{rtl}$$

ANOVA with Dummy Coded Variables

- The ANOVA model can also be re-written using two dummy-coded variables D_{rt} and D_{rl}
  - Becomes a linear model (i.e., regression model)

- D_{rt}
  - $D_{rt}=0$ for respondents in ***cold temperature*** condition
  - $D_{rt}=1$ for respondents in ***warm temperature*** condition

- D_{light}
  - $D_{rl}=0$ for respondents in ***low lighting*** condition
  - $D_{rl}=1$ for respondents in ***high lighting*** condition

ANOVA with Dummy Coded Variables

- The ANOVA model then becomes:

	$D_{rl} = 0$ Low Lighting	$D_{rl} = 1$ High Lighting
$D_{rt} = 0$ Cold Temperature	$\bar{Y}_{Cold,Low}$	$\bar{Y}_{Cold,High}$
$D_{rt} = 1$ Warm Temperature	$\bar{Y}_{Warm,Low}$	$\bar{Y}_{Warm,High}$

$$Y_{rtl} = \beta_0 + \beta_t D_{rt} + \beta_l D_{rl} + \beta_{t*l} D_{rt} D_{rl} + e_{itl}$$

ANOVA Effects Explained

$$Y_{rtl} = \beta_0 + \beta_t D_{rt} + \beta_l D_{rl} + \beta_{t*l} D_{rt} D_{rl} + e_{itl}$$

- β_0 is the mean for the cold and low light condition (reference group)
  - The intercept
- β_t is the change of the mean when comparing cold to warm temperature for a business with low lights (Simple Main Effect)
- β_l is the change of the mean when comparing low to high lights for a business with a cold temperature (Simple Main Effect)
- β_{t*l} is additional mean change that is not explained by the shift in temperature and shift and lights, when both occur (2-Way Interaction)
- Respondents from in the same condition have the same predicted value

ANOVA and the LCDM

- The ANOVA model and the LCDM take the same modeling approach
  - Predict a response using dummy coded variables
    - In LCDM dummy coded variables are latent attributes
  - Using a set of main effects and interactions
    - Links attributes to item response
  - Where possible, we may look for ways to reduce the model
    - Removing non-significant interactions and/or main effects

Differences Between LCDM and ANOVA

- The LCDM and the ANOVA model differ in two ways:
  - Instead of a continuous outcome such as work output the LCDM models a function of the probability of a correct response
    - The logit of a correct response (defined next)
  - Instead of observed "factors" as predictors the LCDM uses discrete *latent* variables (the attributes being measured)
- Attributes are given dummy codes (act as latent factors)
  - α_{ra} = 1 if respondent *r* has **mastered** attribute *a*
  - α_{ra} = 0 if respondent *r* has **not mastered** attribute *a*

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LOGITS EXPLAINED

Model Background

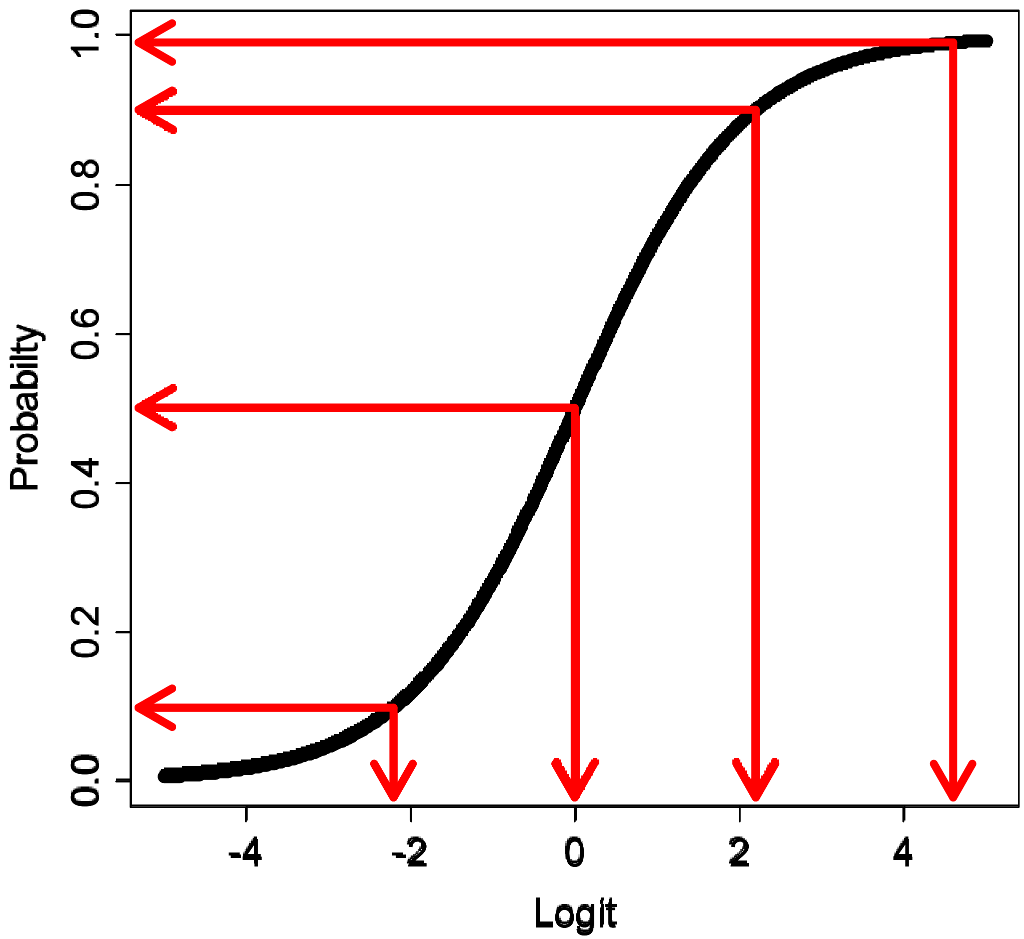
- Just as in IRT models, the LCDM models the ***log-odds*** of a correct response conditional on a respondent's attribute pattern $\boldsymbol{\alpha}_r$
  - The log-odds is called a ***logit***

$$\text{Logit}\left(X_{ri} = 1 \mid \boldsymbol{\alpha}_r\right) = \ln\left(\frac{P\left(X_{ri} = 1 \mid \boldsymbol{\alpha}_r\right)}{1 - P\left(X_{ri} = 1 \mid \boldsymbol{\alpha}_r\right)}\right)$$

- The logit is used because the responses are binary
  - Items are either answered correctly (1) or incorrectly (0)

- The linear model is inappropriate for categorical data
  - Can lead to impossible predictions (i.e., probabilities greater than 1 or less than 0)

More on Logits



Probability	Logit
0.5	0.0
0.9	2.2
0.1	-2.2
0.99	4.6

From Logits to Probabilities

- Whereas logits are useful as the are unbounded continuous variables, categorical data analyses rely on estimated probabilities

- The inverse logit function coverts the unbounded logit to a probability
  - This is also the form of an IRT model (and logistic regression)

$$P\left(X_{ri}=1 \mid \boldsymbol{\alpha}_r\right)=\frac{\exp\left(\mathrm{Logit}\left(X_{ri}=1 \mid \boldsymbol{\alpha}_r\right)\right)}{1+\exp\left(\mathrm{Logit}\left(X_{ri}=1 \mid \boldsymbol{\alpha}_r\right)\right)}$$

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THE LCDM

Building the LCDM

- To demonstrate the LCDM, consider the item 2+3-1=?
  from our basic math example
  - The item measured addition (attribute 1) and subtraction (attribute 2)

- Only attributes defined by the Q-matrix are modeled for an item

- The LCDM provides the logit of a correct response as a function of the latent attributes mastered by a respondent:

$$\operatorname{logit}\left(X_{ri}=1 \mid \boldsymbol{\alpha}_r\right)=\lambda_{i,0}+\lambda_{i,1,(1)}\alpha_{r1}+\lambda_{i,1,(2)}\alpha_{r2}+\lambda_{i,2,(1,2)}\alpha_{r1}\alpha_{r2}$$

LCDM Explained

$$\boxed{\text{logit}\left(X_{ri}=1\,|\,\boldsymbol{\alpha}_r\right)}=\boxed{\lambda_{i,0}}+\lambda_{i,1,(1)}\alpha_{r1}+\lambda_{i,1,(2)}\alpha_{r2}+\lambda_{i,2,(1,2)}\alpha_{r1}\alpha_{r2}$$

- logit(X_{ri} = 1) is the logit of a correct response to item *i* by respondent *r*

- $\lambda_{i,0}$ is the intercept
  - The logit for non-masters of addition and subtraction
  - The reference group is respondents who have not mastered ***either*** attribute (α_{r1} = 0 and α_{r2} = 0)

LCDM Explained

$$\text{logit}\left(X_{ri}=1\,|\,\boldsymbol{\alpha}_r\right)=\lambda_{i,0}+\boxed{\lambda_{i,1,(1)}}\alpha_{r1}+\boxed{\lambda_{i,1,(2)}}\alpha_{r2}+\boxed{\lambda_{i,2,(1,2)}}\alpha_{r1}\alpha_{r2}$$

- $\lambda_{i,1,(1)}$ = **main effect** for addition (attribute 1)
  - The increase in the logit for mastering addition (in someone who has not also mastered subtraction)
- $\lambda_{i,1,(2)}$ = **main effect** for subtraction (attribute 2)
  - The increase in the logit for mastering subtraction (in someone who has not also mastered addition)
- $\lambda_{i,2,(1,2)}$ is the **interaction** between addition and subtraction (attributes 1 and 2)
  - Change in the logit for mastering ***both*** addition & subtraction

Understanding LCDM Notation

- The LCDM item parameters have several subscripts:

$$\lambda_{\boxed{i},\boxed{e},\boxed{(a_1,\ldots)}}$$

- Subscript #1 – *i*: the item to which parameters belong
- Subscript #2 – *e*: the level of the effect
  - 0 is the intercept
  - 1 is the main effect
  - 2 is the two-way interaction
  - 3 is the three-way interaction
- Subscript #3 – (a_1,…): the attributes to which the effect applies
  - Same number of attributes listed as number in Subscript #2

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LCDM: A NUMERICAL EXAMPLE

LCDM with Example Numbers

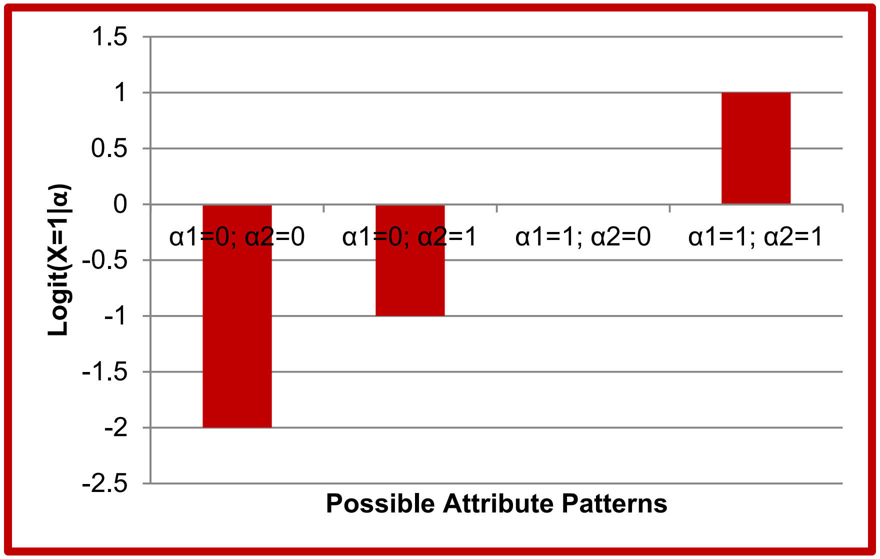
- Imagine we obtained the following estimates for the simple math item:

Parameter	Estimate	Effect Name
$\lambda_{i,0}$	-2	Intercept
$\lambda_{i,1,(1)}$	2	Addition Simple Main Effect
$\lambda_{i,1,(2)}$	1	Subtraction Simple Main Effect
$\lambda_{i,2,(1,2)}$	0	Addition/Subtraction Interaction

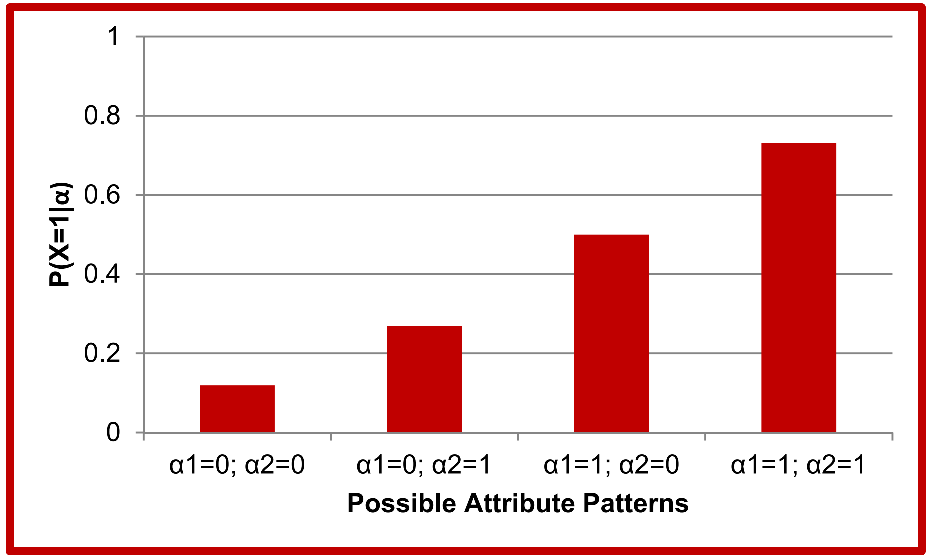
LCDM Predicted Logits and Probabilities

α_1	α_2	LCDM Logit Function	Logit	Probability
0	0	$\lambda_{i,0} + \lambda_{i,1,(1)}*(0) + \lambda_{i,1,(2)}*(0) + \lambda_{i,2,(1,2)}*(0)*(0)$	-2	0.12
0	1	$\lambda_{i,0} + \lambda_{i,1,(1)}*(0) + \lambda_{i,1,(2)}*(1) + \lambda_{i,2,(1,2)}*(0)*(1)$	-1	0.27
1	0	$\lambda_{i,0} + \lambda_{i,1,(1)}*(1) + \lambda_{i,1,(2)}*(0) + \lambda_{i,2,(1,2)}*(1)*(0)$	0	0.50
1	0	$\lambda_{i,0} + \lambda_{i,1,(1)}*(1) + \lambda_{i,1,(2)}*(1) + \lambda_{i,2,(1,2)}*(1)*(1)$	1	0.73

Logit Response Function



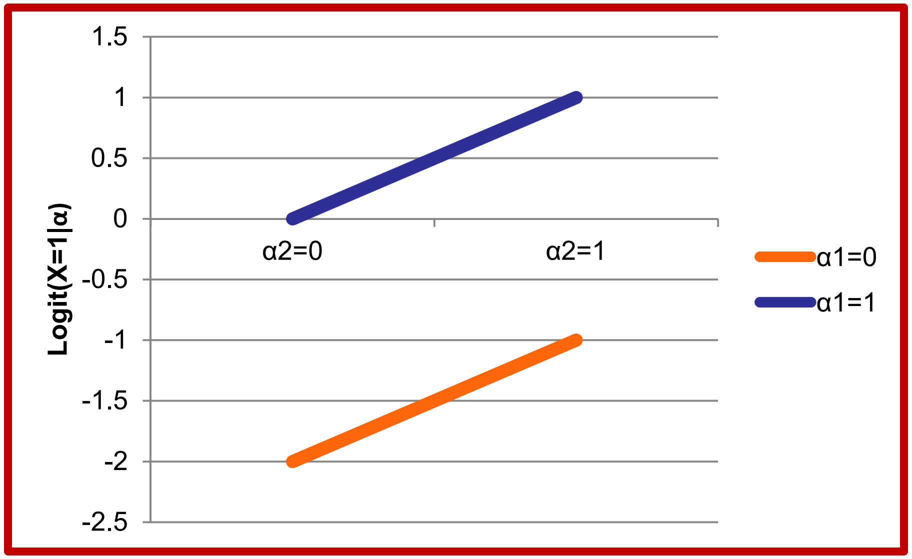
Probability Response Function



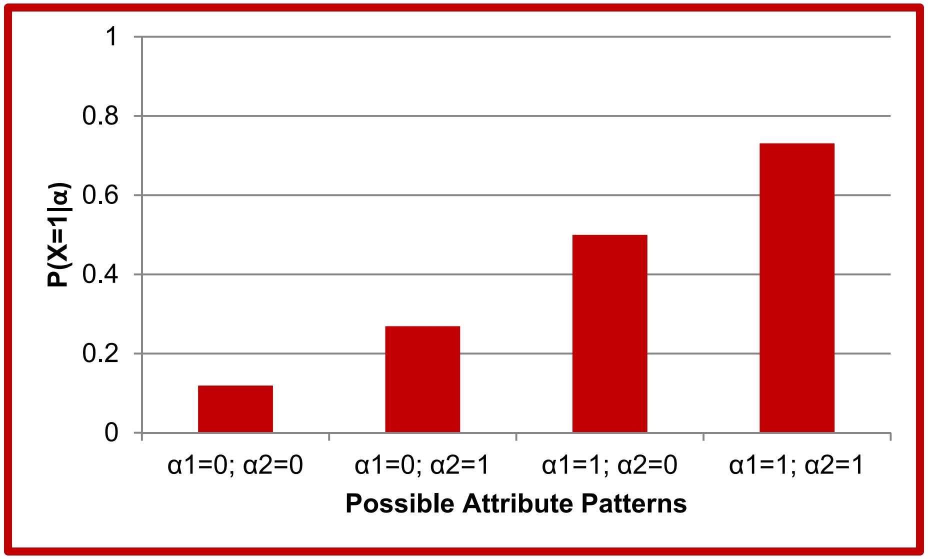
LCDM Interaction Plots

- The LCDM interaction term can be investigated via plots
- **No interaction**: parallel lines for the logit
  - Compensatory RUM (Hartz, 2002)

Logit Response Function



Probability Response Function

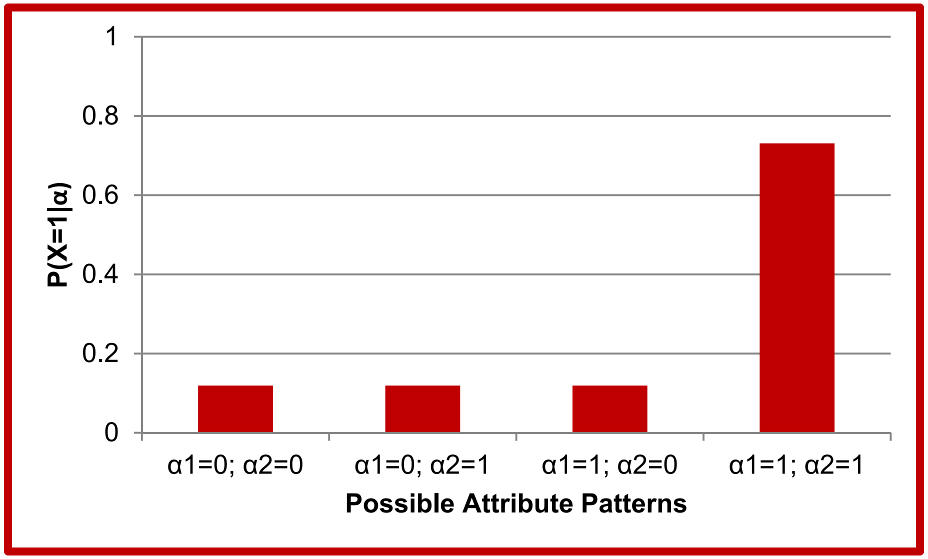


Strong Positive Interactions

- **Positive interaction**: over-additive logit model
  - Conjunctive model (i.e., all-or-none)
  - DINA model (Haertel, 1989; Junker & Sijtsma, 1999)

Logit Response Function

Probability Response Function



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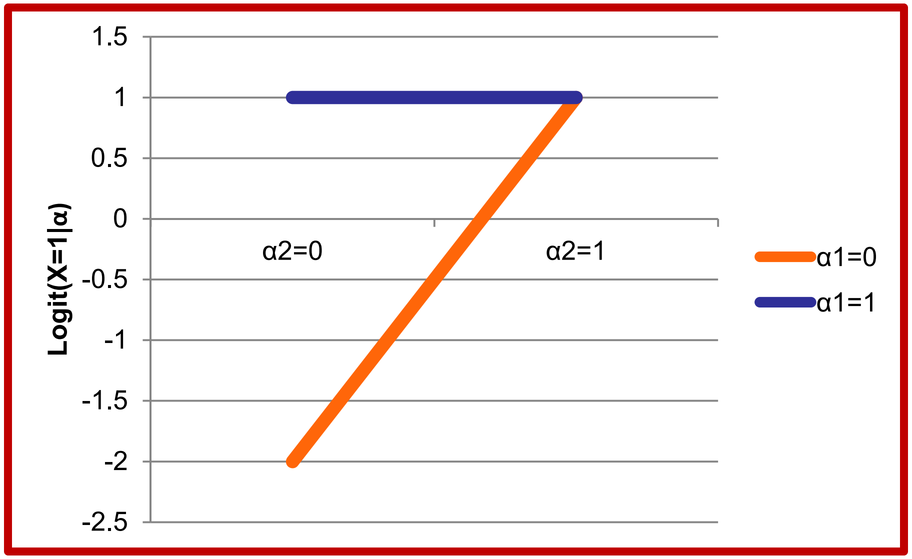
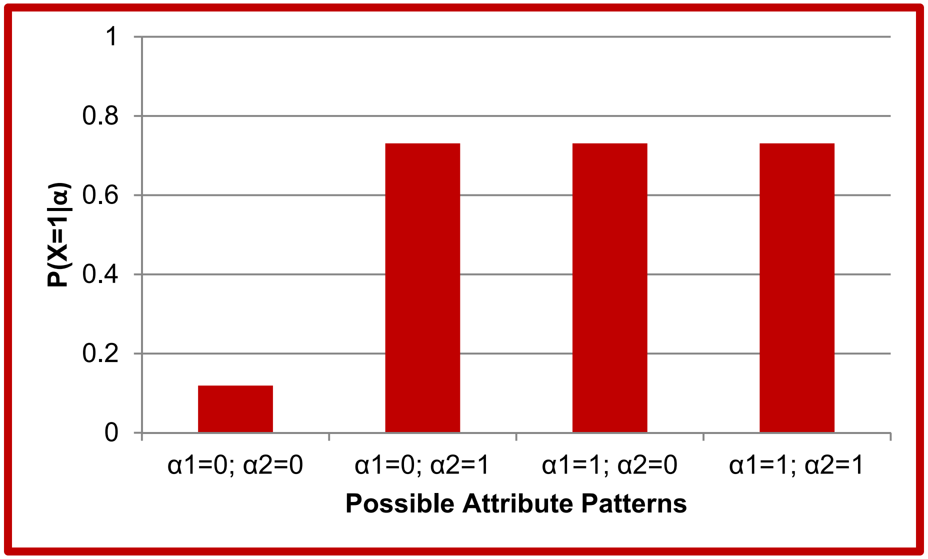
Strong Negative Interactions

- **Negative interaction**: under-additive logit model
  - Disjunctive model (i.e., one-or-more)
  - DINO model (Templin & Henson, 2006)

Logit Response Function



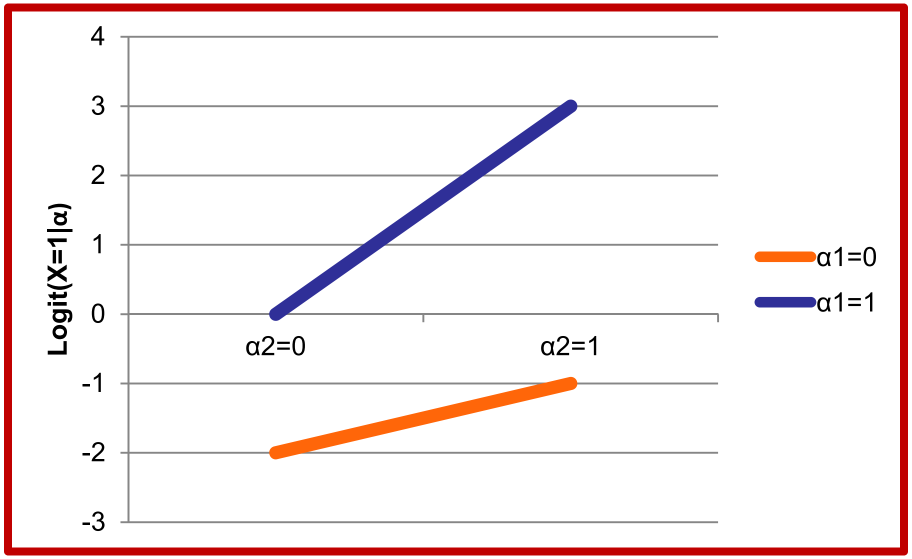
Probability Response Function



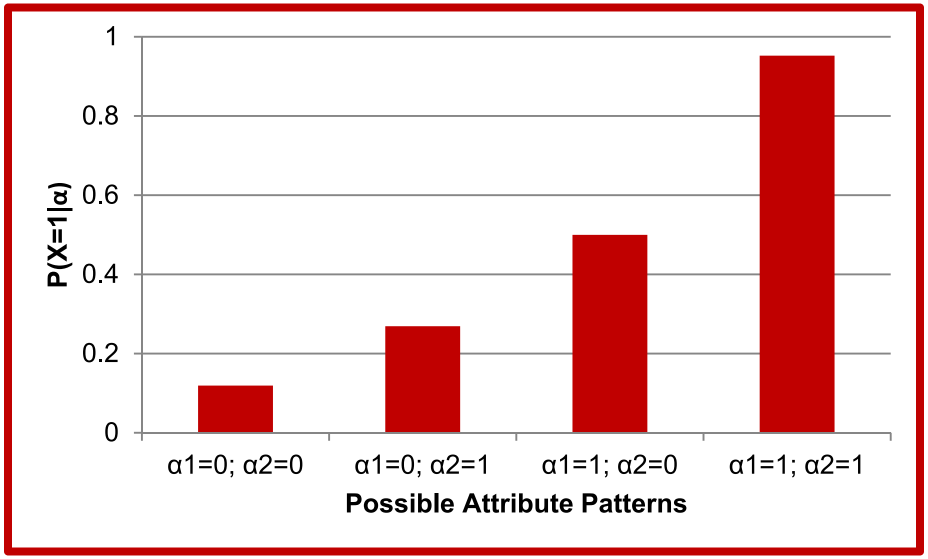
Less Extreme Interactions

- Extreme interactions are unlikely in practice
- Below: positive interaction with positive main effects

Logit Response Function



Probability Response Function



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GENERAL FORM OF THE LCDM

More General Versions of the LCDM

- The LCDM is based on the General Diagnostic Model by von Davier (GDM; 2005)
  - The GDM allows for both categorical and continuous latent variables

- For items measuring more than two attributes, higher level interactions are possible
  - Difficult to estimate in practice

- The LCDM appears in the psychometric literature in a more general form
  - See Henson, Templin, & Willse (2009)

General Form of the LCDM

- The LCDM specifies the probability of a correct response as a function of a set of attributes and a Q-matrix:

$$P(X_{ri} = 1 \mid \boldsymbol{\alpha}_r) = \frac{e^{\boldsymbol{\lambda}_i^T \mathbf{h}(\mathbf{q}_i, \boldsymbol{\alpha}_r)}}{1 + e^{\boldsymbol{\lambda}_i^T \mathbf{h}(\mathbf{q}_i, \boldsymbol{\alpha}_r)}}$$

- The term in the exponent is the logit we have been using all along

Logit(X_{ri} =1|$\boldsymbol{\alpha}_r$)

$$\boldsymbol{\lambda}_i^T \mathbf{h}(\mathbf{q}_i, \boldsymbol{\alpha}_r) = \lambda_{i,0} + \sum_{u=1}^{K} \lambda_{i,1,(u)}(\alpha_{ru} q_{iu}) + \sum_{u=1}^{K} \sum_{v>u} \lambda_{i,2,(u,v)}(\alpha_{ru}\alpha_{rv} q_{iu} q_{iv}) + \ldots$$

Intercepts

Main Effects

Two-Way Interactions

Higher Interactions

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SUBSUMED MODELS

Previously Popular DCMs

- Because the advent of the GDM and LCDM has been fairly recent, other earlier DCMs are still in use

- Such DCMs are much more restrictive than the LCDM
  - Not discussed at length here
  - It is anticipated that field will adapt to more general forms

- Each of these models can be fit using the LCDM
  - Fixing certain model parameters

- Shown for reference purposes
  - See Henson, Templin, & Willse (2009) for more detail

Other DCMs with the LCDM

- The Big 6 - DCMs with latent variables:
  - **DINA** (Deterministic Inputs, Noisy ‘AND’ Gate)
    - Haertel (1989); Junker and Sijtsma (1999)
  - **NIDA** (Noisy Inputs, Deterministic ‘AND’ Gate)
    - Maris (1995)
  - **RUM** (Reparameterized Unified Model)
    - Hartz (2002)
  - **DINO** (Deterministic Inputs, Noisy ‘OR’ Gate)
    - Templin & Henson (2006)
  - **NIDO** (Noisy Inputs, Deterministic ‘OR’ Gate)
    - Templin (2006)
  - **C-RUM** (Compensatory Reparameterized Unified Model)
    - Hartz (2002)

Other DCMs with the LCDM

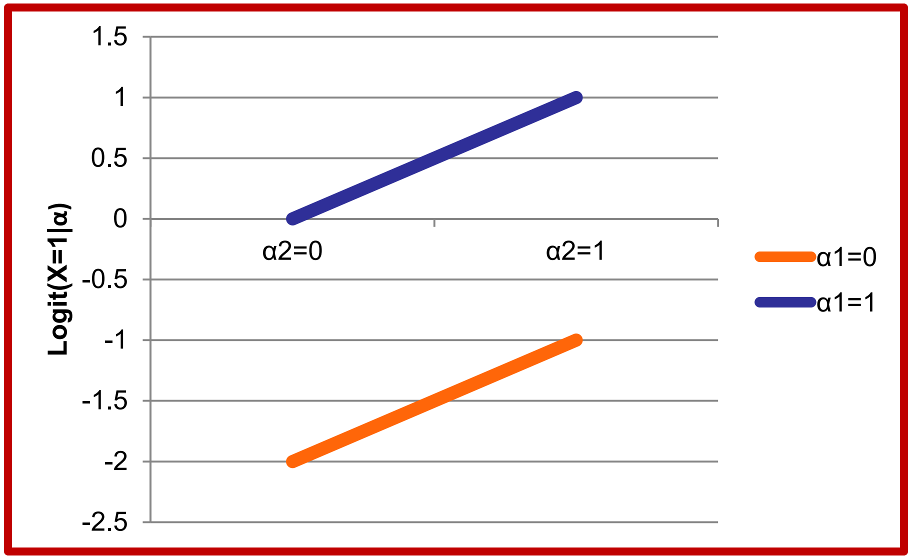
LCDM Parameters	Non-compensatory Models			Compensatory Models		
	DINA	NIDA	NC-RUM	DINO	NIDO	C-RUM
Main Effects	Zero	Positive	Positive	Positive	Positive	Positive
Interactions	Positive	Positive	Positive	Negative	Zero	Zero
Parameter Restrictions	Across Attributes	Across Items	---	Across Attributes	Across Items	---

Adapted from: Rupp, Templin, and Henson (forthcoming, 2010)

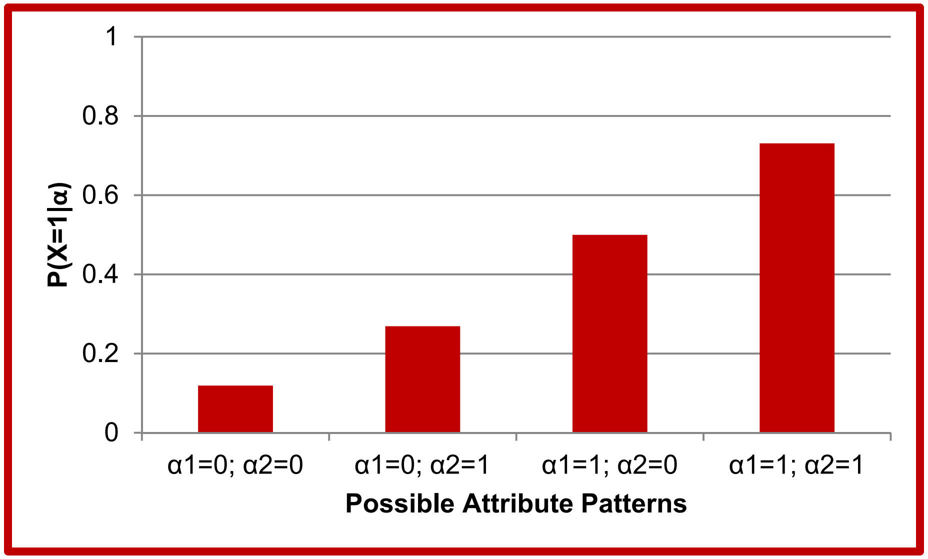
Compensatory RUM (Hartz, 2002)

- No interactions in model
- **No interaction**: parallel lines for the logit

Logit Response Function



Probability Response Function

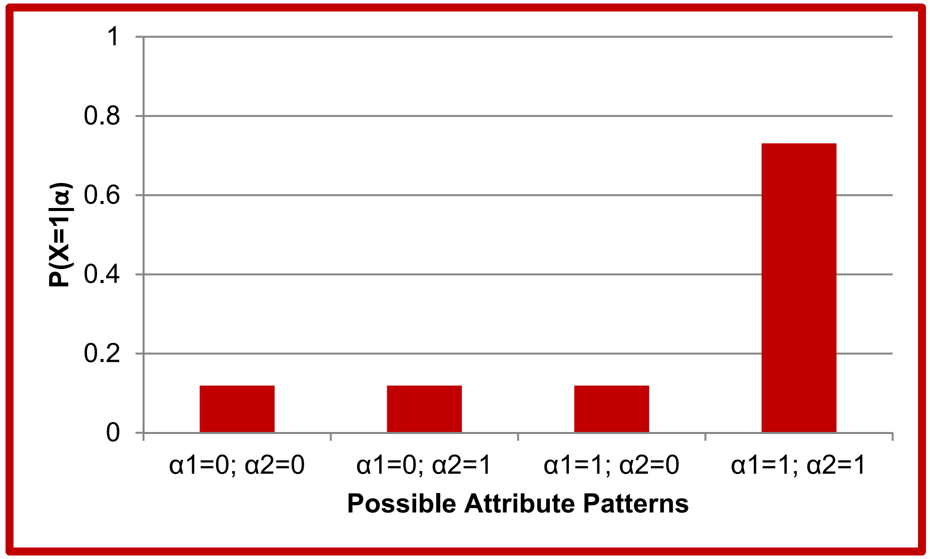


DINA Model (Haertel, 1989; Junker & Sijstma, 1999)

- **Positive interaction**: over-additive logit model
  - Highest interaction parameter is non-zero
  - All main effects (and lower interactions) zero

Logit Response Function

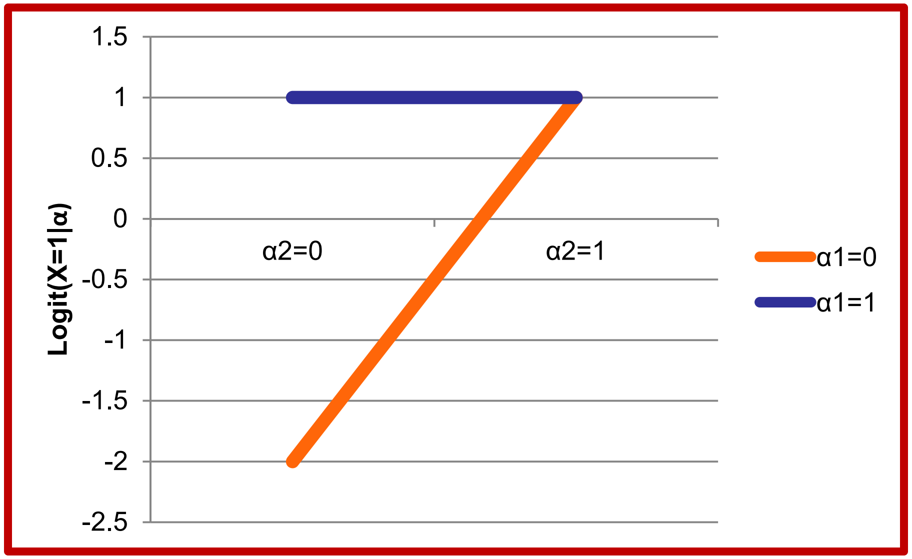
Probability Response Function



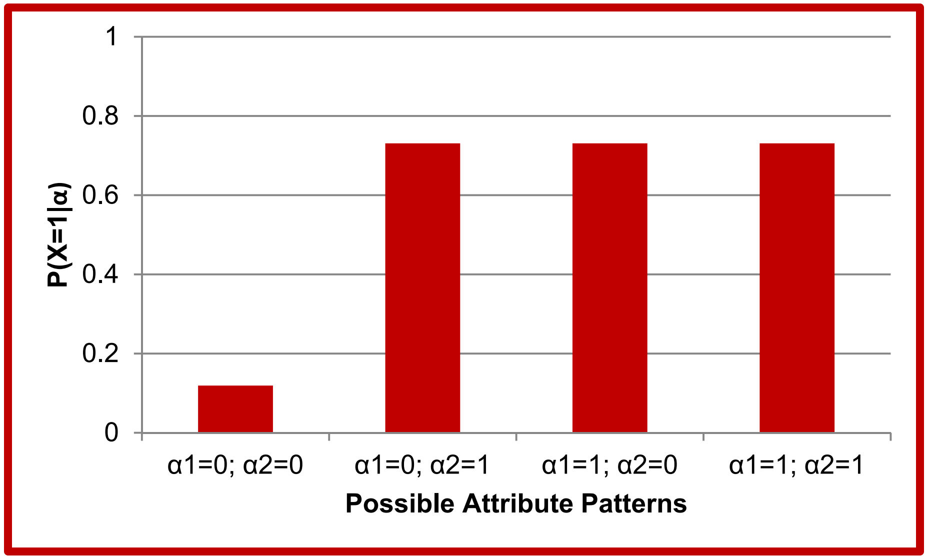
DINO Model (Templin & Henson, 2006)

- **Negative interaction**: under-additive logit model
  - All main effects equal
  - Interaction terms are -1 sum of corresponding lower effects

Logit Response Function



Probability Response Function



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CONCLUDING REMARKS

Session 2 – Take-Home Points

- The LCDM uses an ANOVA-like approach to map latent attributes onto item responses
  - Uses main effects and interactions for each attribute
  - Uses a logit link function

- Multiple diagnostic models are subsumed by the LCDM

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