## Data Example

A simple example is fraction subtraction, analyzed in K. Tatsuoka (1990) and C. Tatsuoka (2002).

Let's consider what one must master to subtract fractions.

# Attributes

According to Mislevy (1996).

(1) performing basic fraction subtraction operation

(2) simplifying/reducing

(3) separating whole number from fraction

(4) borrowing one from whole number to fraction

(5) converting whole number to fraction

## Latent Variable Models

Notation:

$\boldsymbol{Y}$ denotes a vector of binary responses for the $J$ items of an examination.

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, .., \alpha_K)'$. The $k$th element, $\alpha_k$, of $\boldsymbol{\alpha}$ is a binary indicator of an examinee's classification with regard to the $k$th attribute.

Components of $\boldsymbol{Y}$ are independent given $\boldsymbol{\alpha}$.

$\boldsymbol{Q} = [q_{jk}]$, where element $q_{jk}$ of matrix $\boldsymbol{Q}$ indicates whether $k$th skill is required for $j$th item.

# NIDA Model

Let $\eta_{ijk}$ indicate whether the $i$th subject correctly applied the $k$th attribute in completing the $j$th item.

$$s_k = P(\eta_{ijk} = 0 \mid \alpha_{ik} = 1, q_{jk} = 1)$$

$$g_k = P(\eta_{ijk} = 1 \mid \alpha_{ik} = 0, q_{jk} = 1)$$

$$Y_{ij} = \prod_{k=1}^{K} \eta_{ijk}$$

$$P(Y_{ij} = 1 \mid \boldsymbol{\alpha}_i, \boldsymbol{s}, \boldsymbol{g}) = \prod_{k=1}^{K} P(\eta_{ijk} = 1 \mid \alpha_{ik}, s_k, g_k)$$

$$= \prod_{k=1}^{K} \left[ (1 - s_k)^{\alpha_{ik}} g_k^{1-\alpha_{ik}} \right]^{q_{jk}} .$$

$$L(\boldsymbol{s}, \boldsymbol{g}; \boldsymbol{\alpha}) =$$

$$\prod_{i=1}^{N} \prod_{j=1}^{J} \left\{ \prod_{k=1}^{K} \left[ (1 - s_k)^{\alpha_{ik}} g_k^{1-\alpha_{ik}} \right]^{q_{jk}} \right\}^{Y_{ij}} \left\{ 1 - \prod_{k=1}^{K} \left[ (1 - s_k)^{\alpha_{ik}} g_k^{1-\alpha_{ik}} \right]^{q_{jk}} \right\}^{1-Y_{ij}} .$$

# The DINA Model

Let $\eta_{ij}$ denote whether the $i$th examinee possesses the attributes required for the $j$th item.

$$\eta_{ij} = \prod_{k=1}^{K} \alpha_{ik}^{q_{jk}}.$$

The parameter $s_j$ refers to the probability of slipping and incorrectly answering the item when $\eta_{ij} = 1$, and $g_j$ is the probability of correctly guessing the answer when $\eta_{ij} = 0$.

$$P(Y_{ij} = 1 \mid \boldsymbol{\alpha}) = (1 - s_j)^{\eta_{ij}} g_j^{1-\eta_{ij}}.$$

$$L(\boldsymbol{s}, \boldsymbol{g}; \boldsymbol{\alpha}) =$$

$$\prod_{i=1}^{N} \prod_{j=1}^{J} \left[(1 - s_j)^{y_{ij}} s_j^{1-y_{ij}}\right]^{\eta_{ij}} \left[g_j^{y_{ij}} (1 - g_j)^{1-y_{ij}}\right]^{1-\eta_{ij}}.$$

# The Fusion Model

$$P(Y_{ij} = 1 | \boldsymbol{\alpha}, \pi_j^*, r_{jk}^*, c_j, \eta_i)$$

$$= \pi_j^* P_{c_j}(\eta_i) \prod_{k=1}^{K} r_{jk}^{*(1-\alpha_{ik})q_{jk}}$$

$P_{c_j}(\eta)$ is a Rasch IRF with parameter $c_j$

Let $Y_{ijk}$ be a latent response denoting whether examinee $i$ correctly applies a required attribute $k$ to item $j$.

$$\pi_j^* = \prod_{k=1}^{K} P[Y_{ijk} = 1 | \alpha_{ik} = 1]$$

$$r_{jk}^* = P[Y_{ijk} = 1 | \alpha_{ik} = 0] / P[Y_{ijk} = 1 | \alpha_{ik} = 1]$$