Transitioning from Repeated Measures ANOVA to Mixed-Models

ERSH 8310

The University of Georgia

Today's Class

- An introduction to modern versions of repeated measures ANOVA
  - Mixed models

- Mixed models are powerful tools for a lot of research applications:
  - Repeated measures
  - Longitudinal data (i.e, growth models)
  - Clustered data (i.e., hierarchical linear models)

- If you stare at them long enough you get even more statistical methods out of mixed models:
  - Confirmatory Factor Analysis/Structural Equation Modeling
  - Item Response Models

- Plus, mixed models operate using a estimation techniques that allow for missing data

Today's Data Set – Same as Before

- Consider an experiment in which college students search for a particular letter in a string of letters on a computer screen
  - Half of the time the letter occurs in the string, and half of the time it does not
- On one third of the trials the letter string is a word (condition a_1), on one third it is a pronounceable nonword (a_2), and on one third it is an unpronounceable set of random letters (a_3)
- The response measure is the average speed with which subjects correctly detect the target letter, measured in milliseconds
- The experiment is a single-factor AxS design with a=3 types of letter strings

Our Design Setup

Table 16.1: *The notation system for an $A \times S$ design with $a = 4$ conditions and $n = 3$ subjects.*

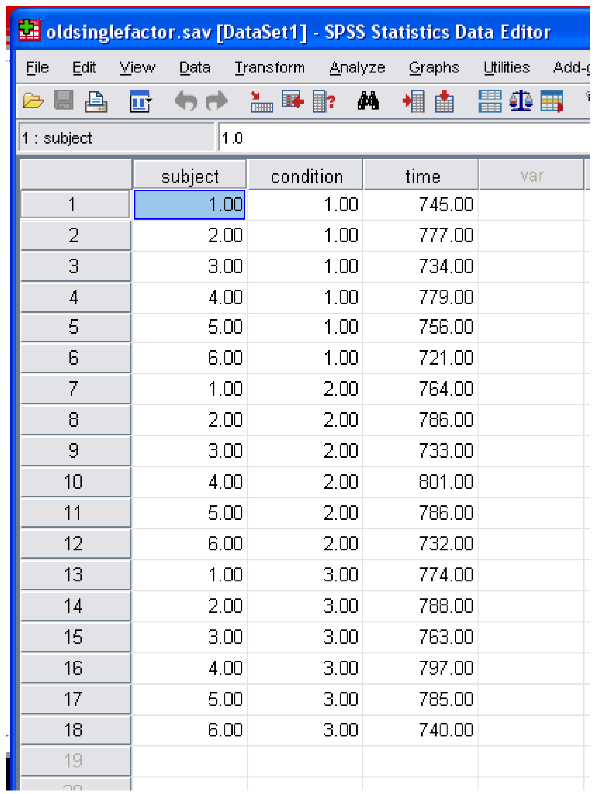
	Levels of factor A				
Subjects	a_1	a_2	a_3	a_4	Sum
s_1	Y_{11}	Y_{12}	Y_{13}	Y_{14}	S_1
s_2	Y_{21}	Y_{22}	Y_{23}	Y_{24}	S_2
s_3	Y_{31}	Y_{32}	Y_{33}	Y_{34}	S_3
Sum	A_1	A_2	A_3	A_4	T

The Old Style of Analysis

- Prior to running the repeated measures analysis, let's imagine we tried using what we already know about ANOVA:
  - Let's run the analysis as if we have subjects nested within each factor (not crossed)

The Data Looks Like…

- This is what we call long data
- Every dependent variable value has a row


oldsinglefactor.sav [DataSet1] - SPSS Statistics Data Editor

File Edit View Data Transform Analyze Graphs Utilities Add-

1 : subject 1.0

	subject	condition	time	var
1	1.00	1.00	745.00	
2	2.00	1.00	777.00	
3	3.00	1.00	734.00	
4	4.00	1.00	779.00	
5	5.00	1.00	756.00	
6	6.00	1.00	721.00	
7	1.00	2.00	764.00	
8	2.00	2.00	786.00	
9	3.00	2.00	733.00	
10	4.00	2.00	801.00	
11	5.00	2.00	786.00	
12	6.00	2.00	732.00	
13	1.00	3.00	774.00	
14	2.00	3.00	788.00	
15	3.00	3.00	763.00	
16	4.00	3.00	797.00	
17	5.00	3.00	785.00	
18	6.00	3.00	740.00	
19				

Old Analysis – No Repeated Measures

Between-Subjects Factors

		N
condition	1.00	6
	2.00	6
	3.00	6

Tests of Between-Subjects Effects

Dependent Variable: time

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1575.000[a]	2	787.500	1.299	.302
Intercept	10520284.5	1	10520284.50	17353.524	.000
condition	1575.000	2	787.500	1.299	.302
Error	9093.500	15	606.233		
Total	10530953.0	18			
Corrected Total	10668.500	17			

a. R Squared = .148 (Adjusted R Squared = .034)

Our Statistical Model

- Recall our basic one-way ANOVA statistical model:

$$Y_{ij} = \mu_T + \alpha_j + E_{ij}$$

- Likely the key assumption of the model was independence of error terms
  - Caged as independence of observations

- Error was assumed have a variance of σ^2_{error}

- For our repeated measures analysis, however, we get three observations per person
  - People's scores are highly correlated

Model Implied Covariance Matrix

- Within a subject, the ANOVA model assumes the following covariance matrix
  - Shown for our three repeated observations example
  - A covariance matrix is an un-standardized correlation matrix

$$\mathbf{R} = \begin{bmatrix} \sigma^2_{error} & 0 & 0 \\ 0 & \sigma^2_{error} & 0 \\ 0 & 0 & \sigma^2_{error} \end{bmatrix} \rightarrow \mathbf{V} = \begin{bmatrix} \sigma^2_{error} & 0 & 0 \\ 0 & \sigma^2_{error} & 0 \\ 0 & 0 & \sigma^2_{error} \end{bmatrix}$$

  - The variances are all equal (homogeneity of variance)
  - A zero covariance means a pair of observations are independent (the ANOVA independence assumption)

Remember These Numbers

- From our ANOVA table treating the observations as not being repeated:

$$F_A = 1.299; p = 0.302$$

$$\hat{\sigma}^2_{error} = 606.233$$

- Which implies

$$\hat{\mathbf{R}} = \begin{bmatrix} 606.233 & 0 & 0 \\ 0 & 606.233 & 0 \\ 0 & 0 & 606.233 \end{bmatrix} \rightarrow \hat{\mathbf{V}} = \begin{bmatrix} 606.233 & 0 & 0 \\ 0 & 606.233 & 0 \\ 0 & 0 & 606.233 \end{bmatrix}$$

Then There Was Repeated Measures ANOVA

- Next, we had repeated measures ANOVA…

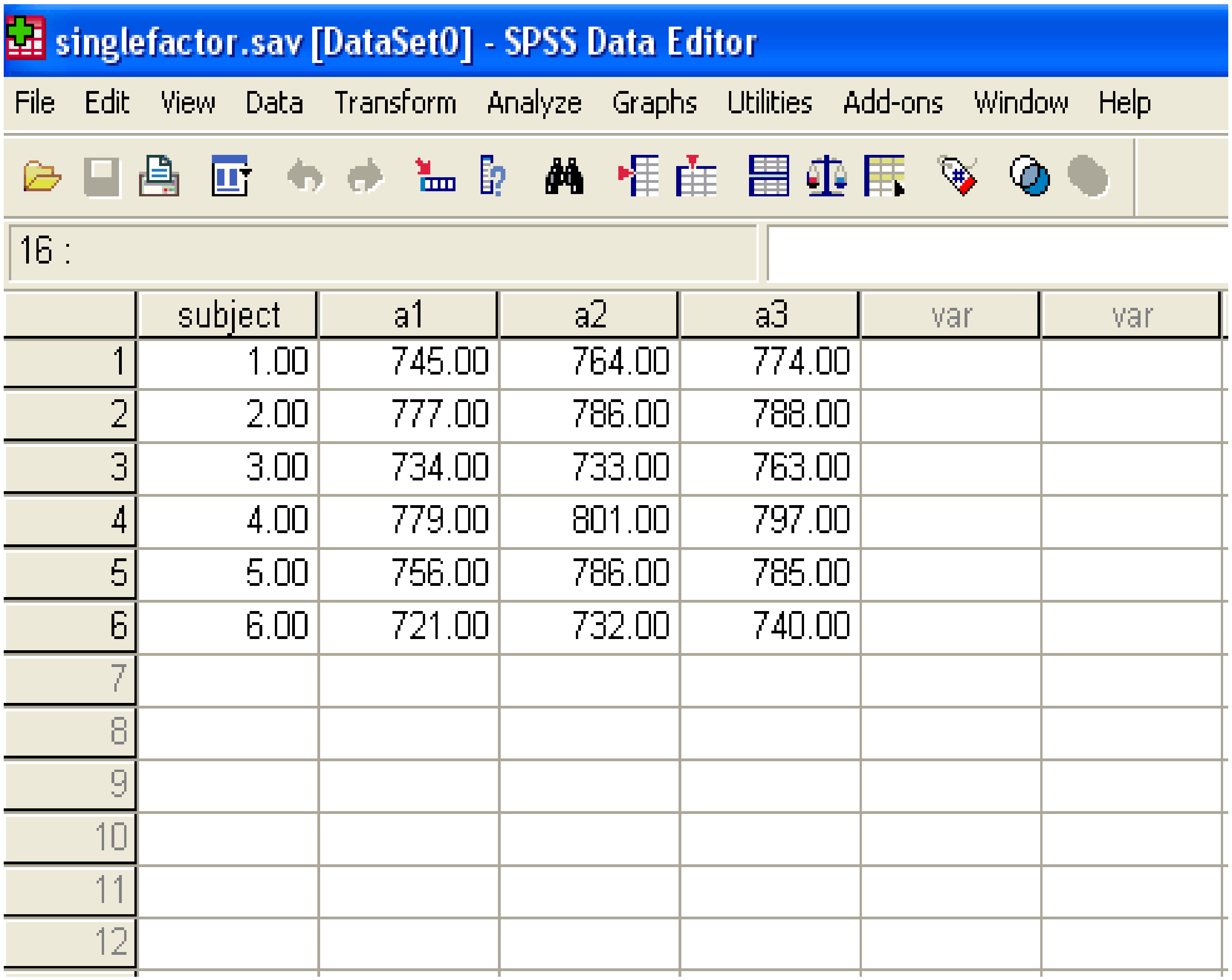
$$Y_{ij} = \mu_T + \alpha_j + S_i + (\alpha S)_{ij} + E_{ij}$$

- However, because of our design (one observation within cell per person), the person by factor interaction was the error term, leaving:

$$Y_{ij} = \mu_T + \alpha_j + S_i + E_{ij}$$

- It assumed a weaker form of compound symmetry (called sphericity) for the error terms
  - All errors had the same variance
  - All covariances were the same

Repeated Measures: Data File in SPSS



	subject	a1	a2	a3	var	var
1	1.00	745.00	764.00	774.00		
2	2.00	777.00	786.00	788.00		
3	3.00	734.00	733.00	763.00		
4	4.00	779.00	801.00	797.00		
5	5.00	756.00	786.00	785.00		
6	6.00	721.00	732.00	740.00		
7						
8						
9						
10						
11						
12						

The data are arranged in a wide-format:
- One variable per column.

Computational Formulas

- The computational formulas for the A ×S design are presented in Table 16.2
  - Note that df_T = an - 1 that is one less than the total number of observations

Table 16.2: Computational formulas for the $A \times S$ design.

Formulas for the analysis of variance

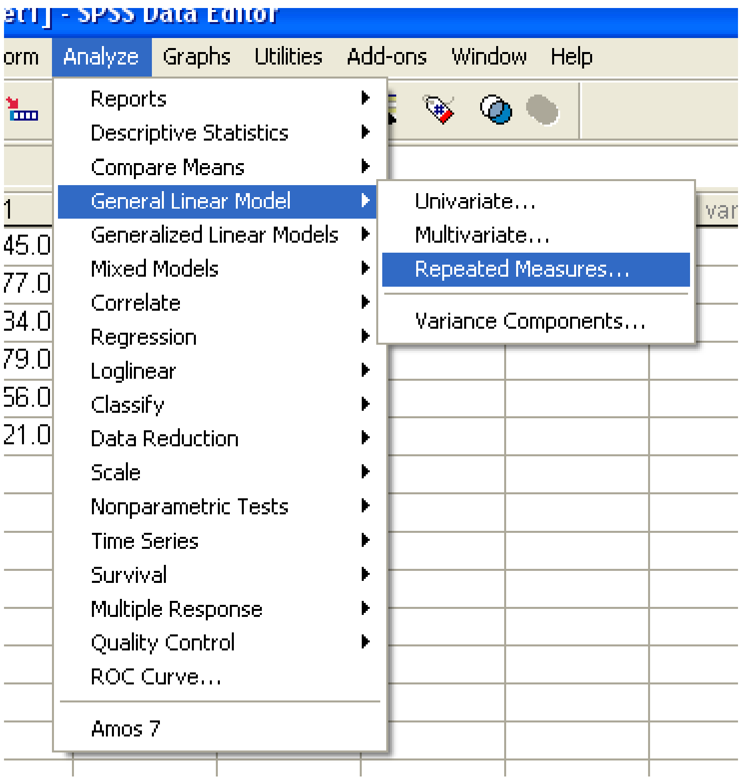
Source	df	SS	MS	F
A	$a-1$	$[A]-[T]$	$\frac{SS_A}{df_A}$	$\frac{MS_A}{MS_{A\times S}}$
S	$n-1$	$[S]-[T]$	$\frac{SS_S}{df_S}$	
$A\times S$	$(a-1)(n-1)$	$[Y]-[A]-[S]+[T]$	$\frac{SS_{A\times S}}{df_{A\times S}}$	
Total	$an-1$	$[Y]-[T]$		

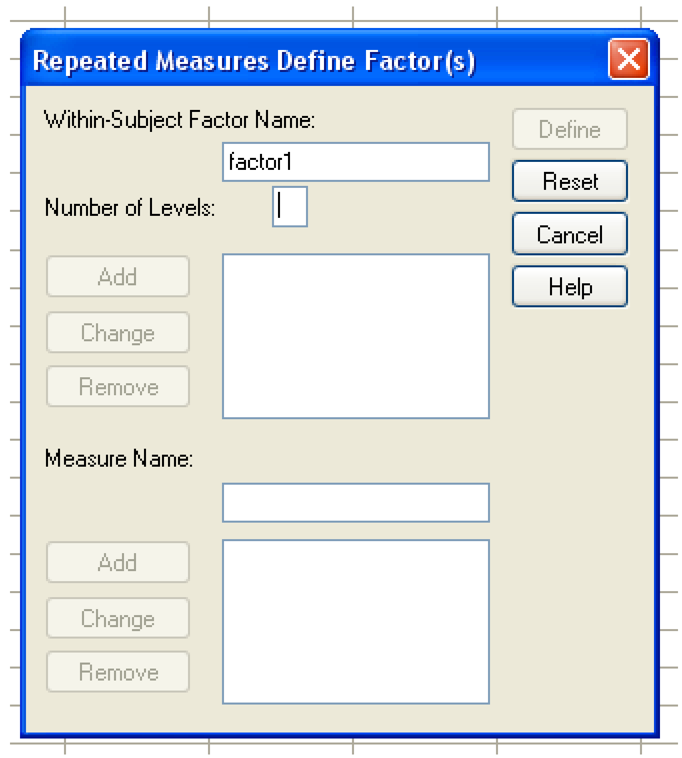
Formulas for the bracket terms

$$[A] = \frac{\sum A_j^2}{n} = n\sum \bar{Y}_A^2 \qquad [S] = \frac{\sum S_i^2}{a} = a\sum \bar{Y}_S^2$$

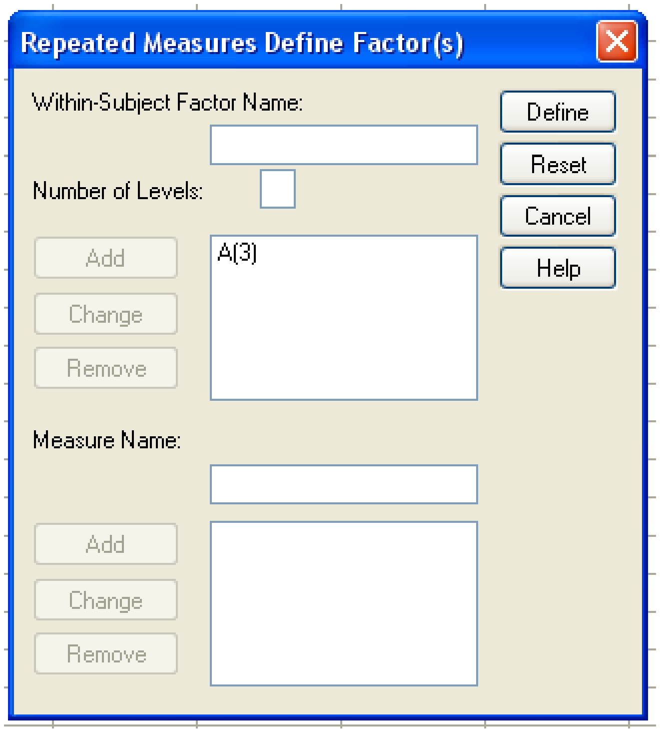
$$[Y] = \sum Y_{ij}^2 \qquad [T] = \frac{T^2}{an} = an\bar{Y}_T^2$$

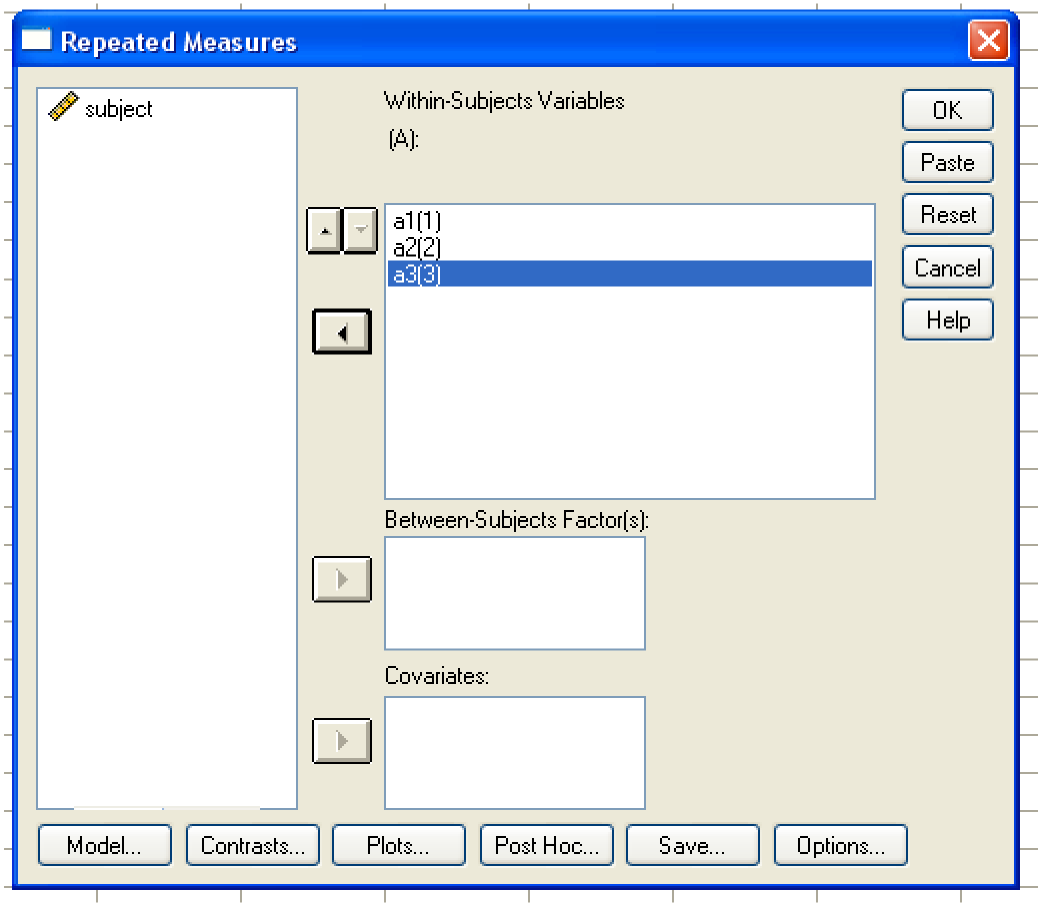
Running the Analysis





Running the Analysis





Analysis Output: SPSS

- There are two relevant parts to SPSS- a test for the factor

Tests of Within-Subjects Effects

Measure:MEASURE_1

Source		Type III Sum of Squares	df	Mean Square	F	Sig.
factor1	Sphericity Assumed	1575.000	2	787.500	14.432	.001
	Greenhouse-Geisser	1575.000	1.610	978.061	14.432	.003
	Huynh-Feldt	1575.000	2.000	787.500	14.432	.001
	Lower-bound	1575.000	1.000	1575.000	14.432	.013
Error(factor1)	Sphericity Assumed	545.667	10	54.567		
	Greenhouse-Geisser	545.667	8.052	67.771		
	Huynh-Feldt	545.667	10.000	54.567		
	Lower-bound	545.667	5.000	109.133		

- And:

Tests of Between-Subjects Effects

Measure:MEASURE_1
Transformed Variable:Average

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Intercept	1.052E7	1	1.052E7	6153.773	.000
Error	8547.833	5	1709.567		

Analysis Output: Table 16.3

The AS data table

Subjects	a_1	a_2	a_3	Sum
	Types of strings			
s_1	745	764	774	2,283
s_2	777	786	788	2,351
s_3	734	733	763	2,230
s_4	779	801	797	2,377
s_5	756	786	785	2,327
s_6	721	732	740	2,193
Sum	4,512	4,602	4,647	13,761
$\sum_i Y_{ij}^2$	3,395,728	3,534,002	3,601,223	
Mean	752.00	767.00	774.50	
s_j	23.26	29.22	20.60	
s_{M_j}	9.50	11.93	8.41	

Calculation of the bracket terms

$$[A] = \frac{\sum A_j^2}{n} = \frac{4{,}512^2 + 4{,}602^2 + 4{,}647^2}{6} = 10{,}521{,}859.5$$

$$[S] = \frac{\sum S_i^2}{a} = \frac{2{,}283^2 + \cdots + 2{,}193^2}{3} = 10{,}528{,}832.3$$

$$[Y] = \sum Y_{ij}^2 = 745^2 + 764^2 + \cdots + 732^2 + 740^2 = 10{,}530{,}953.0$$

$$[T] = \frac{T^2}{an} = \frac{13{,}761^2}{(3)(6)} = 10{,}520{,}284.5$$

Summary of the analysis of variance

Source	SS		df	MS	F
A	$[A] - [T] =$	1,575.0	2	787.50	14.43*
S	$[S] - [T] =$	8,547.8	5	1,709.56	
$A \times S$	$[Y] - [A] - [S] + [T] =$	545.7	10	54.57	
Total	$[Y] - [T] =$	10,668.5	17		

* $p < .05$

Matching Output to Interpretation

Tests of Within-Subjects Effects

Measure:MEASURE_1

Source		Type III Sum of Squares	df	Mean Square	F	Sig.
factor1	Sphericity Assumed	1575.000	2	787.500	14.432	.001
	Greenhouse-Geisser	1575.000	1.610	978.061	14.432	.003
	Huynh-Feldt	1575.000	2.000	787.500	14.432	.001
	Lower-bound	1575.000	1.000	1575.000	14.432	.013
Error(factor1)	Sphericity Assumed	545.667	10	54.567		
	Greenhouse-Geisser	545.667	8.052	67.771		
	Huynh-Feldt	545.667	10.000	54.567		
	Lower-bound	545.667	5.000	109.133		

Source: A

Source: SxA
Now error

Tests of Between-Subjects Effects

Measure:MEASURE_1
Transformed Variable:Average

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Intercept	1.052E7	1	1.052E7	6153.773	.000
Error	8547.833	5	1709.567		

Source: S –
Not typically
interpreted

Remember These Numbers

- From our ANOVA table treating the observations as not being repeated:

$$F_A = 14.432; p = 0.001$$

$$\hat{\sigma}^2_{error} = \hat{\sigma}^2_{AxS} = 54.567$$

- But…what about the original variance of error?
- It is now separated into two components (piles?)…
  - Originally was 606.233
  - Our new term is 54.567, leaving 551.67 behind…
    - The pile left behind now becomes the covariance

Implied Covariance in RM

- Our repeated measures ANOVA thus implies the following:

$$\mathbf{R} = \mathbf{V} = \begin{bmatrix} \sigma^2_{error} & CS & CS \\ CS & \sigma^2_{error} & CS \\ CS & CS & \sigma^2_{error} \end{bmatrix} = \begin{bmatrix} 606.233 & 551.67 & 551.67 \\ 551.67 & 606.233 & 551.67 \\ 551.67 & 551.67 & 606.233 \end{bmatrix}$$

- This is how compound symmetry looks
  - Same values on diagonal
  - Same values on off-diagonal
    - Not zero anymore
- Is this plausible?
  - Maybe yes: non-zero off-diagonals are better than zero
  - Maybe no: not all data have this structure
    - Longitudinal data almost always do not

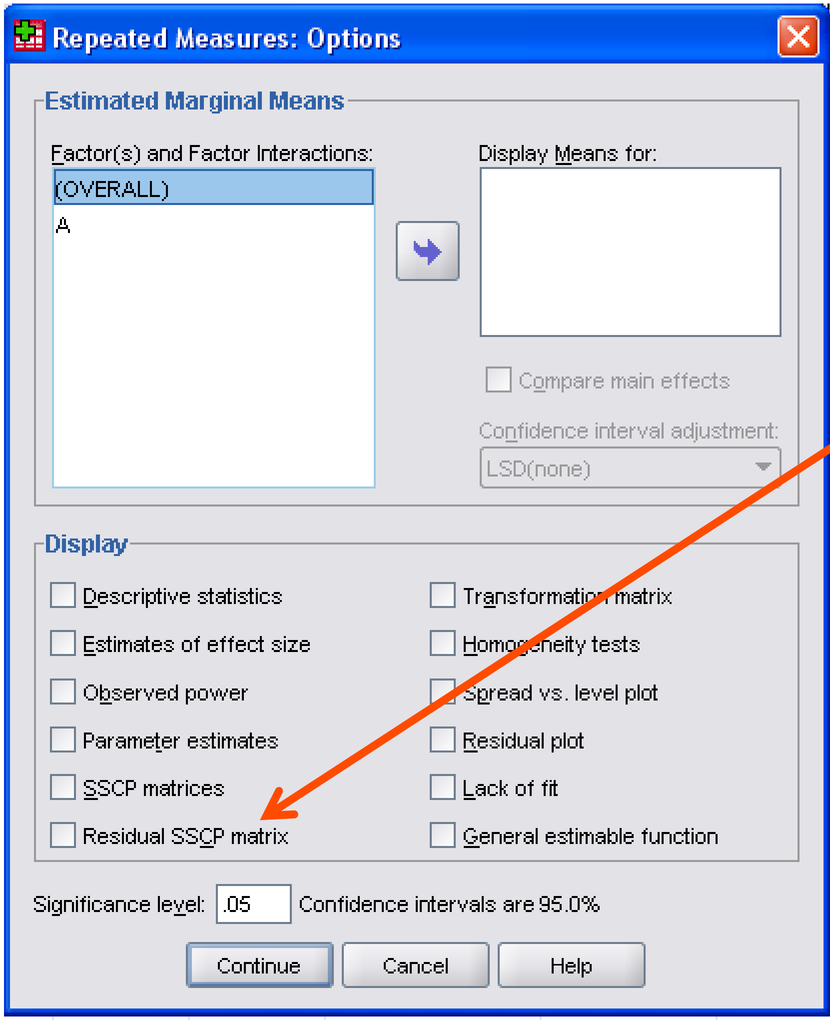
And…Multivariate???

- The book had mentioned another approach, one called the multivariate approach.

- The multivariate approach looks for differences in the three factor levels simultaneously by looking at all data together
  - Uses a matrix-algebra extension of statistics
  - Taught in ERSH 8350

- Estimates every term in the covariance matrix **R**
  - Needs more data
  - Less powerful when compound symmetry holds

Getting Full Covariance Estimates in SPSS



Under Options (in RM ANOVA), check the "Residual SSCP matrix" box

The "Unstructured" (FULL) R Matrix

Residual SSCP Matrix

		a1	a2	a3
Sum-of-Squares and Cross-Products	a1	2704.000	3187.000	2267.000
	a2	3187.000	4268.000	2821.000
	a3	2267.000	2821.000	2121.500
Covariance	a1	540.800	637.400	453.400
	a2	637.400	853.600	564.200
	a3	453.400	564.200	424.300
Correlation	a1	1.000	.938	.947
	a2	.938	1.000	.937
	a3	.947	.937	1.000

Based on Type III Sum of Squares

Note: When using the univariate tests (F), the unstructured **R** matrix is not used for calculations.

It is only used in the Multivariate approach.

$$\mathbf{R} = \mathbf{V} = \begin{bmatrix} \sigma^2_{error_1} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma^2_{error_2} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma^2_{error_3} \end{bmatrix} = \begin{bmatrix} 540.8 & 637.4 & 453.4 \\ 637.4 & 853.6 & 564.2 \\ 453.4 & 564.2 & 424.3 \end{bmatrix}$$

So..Which R Matrix is Right?

Independence	Compound Symmetry	Unstructured
$\begin{bmatrix} \sigma^2_{error} & 0 & 0 \\ 0 & \sigma^2_{error} & 0 \\ 0 & 0 & \sigma^2_{error} \end{bmatrix}$	$\begin{bmatrix} 606.2 & 551.7 & 551.7 \\ 551.7 & 606.2 & 551.7 \\ 551.7 & 551.7 & 606.2 \end{bmatrix}$	$\begin{bmatrix} 540.8 & 637.4 & 453.4 \\ 637.4 & 853.6 & 564.2 \\ 453.4 & 564.2 & 424.3 \end{bmatrix}$
Regular ANOVA	Repeated Measures ANOVA	Multivariate ANOVA

Restrictive Assumptions ⟷ Relaxed Assumptions

Few Parameters ⟷ More Parameters

Enter…the Mixed Model

- Mixed-effects models get their name from the combination of fixed effects (our treatment effects α) and random effects (our subject effects S)

- In their most basic form, they mimic the varying types of ANOVA models
  - Can fit differing structures for R/V matrices

- Using a different method of estimation (likelihood based), they:
  - Give less biased estimates of variances/covariances
  - A measure of which structure is correct (independence, CS, unstructured, etc…)
  - The ability to incorporate missing data directly
    - No need to throw away incomplete cases or impute for missing values

The Mixed Model

- The mixed model looks very similar to ANOVA:

$$Y_{ij} = \mu_T + \alpha_j + S_i + E_{ij}$$

- The difference is our assumptions:
  - μ_T is a "fixed" effect – no estimate of variability
  - α_j are "fixed" effects – no estimate of variability
  - S_i are "random" effects – have normal distribution with zero mean and variance τ^2
    - Called "random intercepts" – each subject has a center point
  - E_{ij} are "random" – as usual
    - Have normal distribution with zero mean and variance σ^2_{error}

The OLD ANOVA Model, Redefined

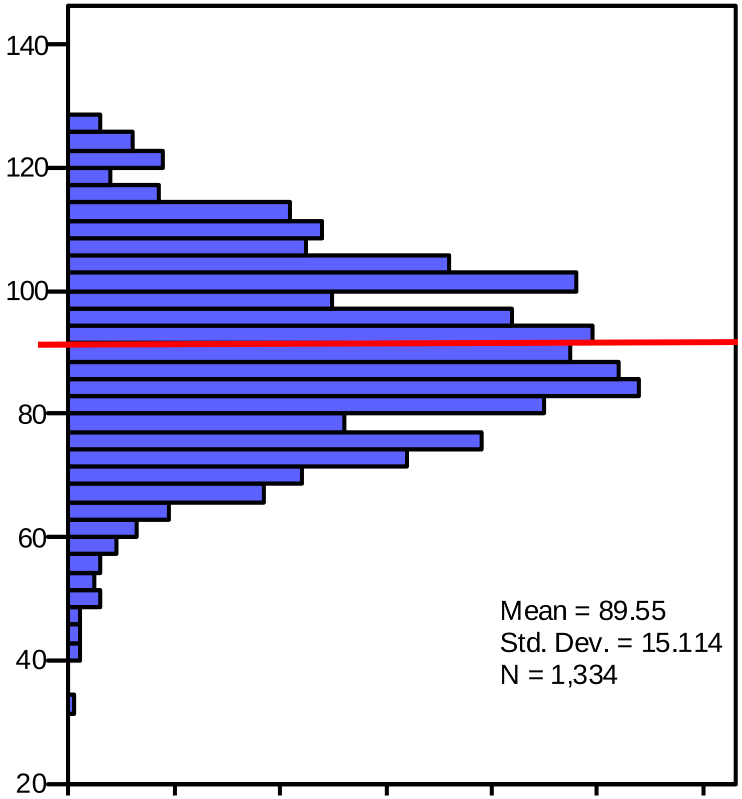
$$Y_{ij} = \boxed{\mu_T + \alpha_j} + \boxed{E_{ij}}$$

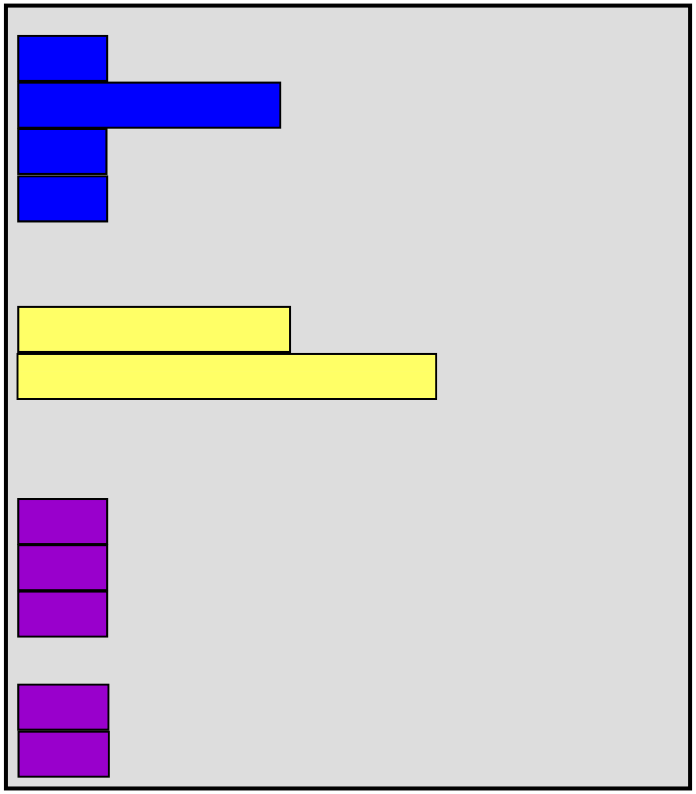
- **Model for the Means (Predicted Values):**
  - Each person's expected (predicted) outcome is a function of his/her treatment group (or values on covariates, and their interactions)
  - IV and DV are each measured only once per person (*i* subscript)

- **Model for the Variance:**
  - $E_{ij} \sim N(0, \sigma_e^2)$ → **ONE** residual (unexplained) deviation
  - E_{ij} has a mean of 0 with some estimated constant variance (σ_e^2), is normally distributed, is unrelated to the IV, and is unrelated across people (across all observations, just people here)

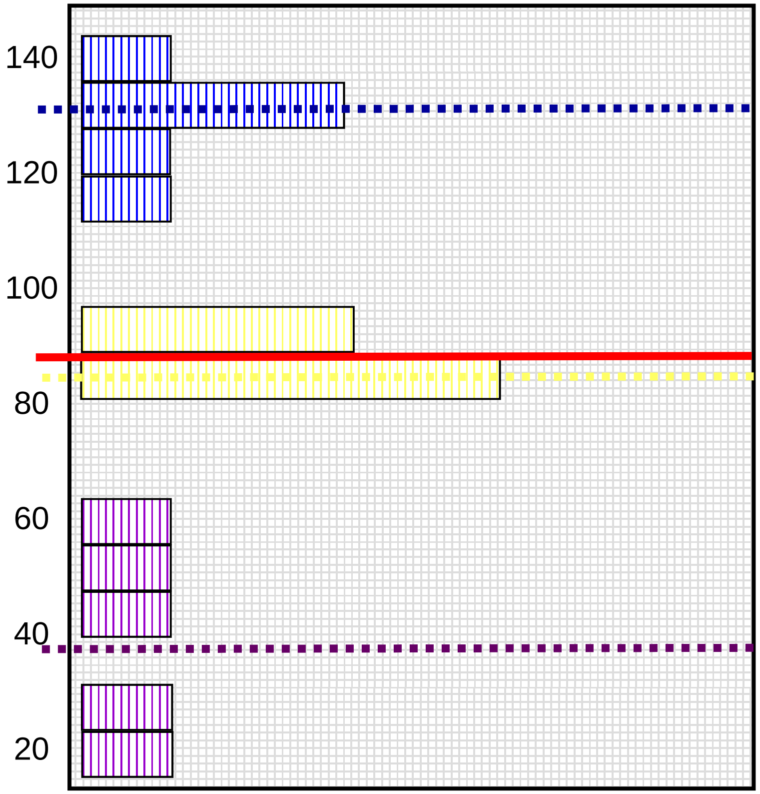
Adding Within-Person Variance to the Model for the Variances

Full Sample Distribution: 3 People, 5 Occasions each





Empty +Within-Person Model



Start off with Mean of Y as "best guess" for any value:

= Grand Mean

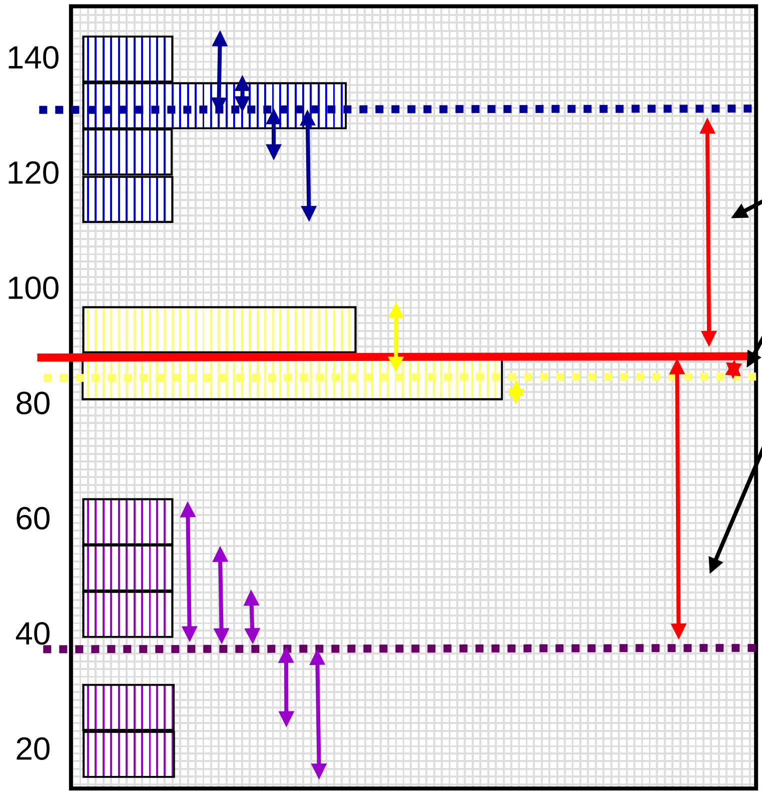
= Fixed Intercept

Can make better guess by taking advantage of repeated observations:

= Person Mean

→ Random Intercept

Empty +Within-Person Model



Variance of Y → 2 sources:

Between-Person Variance:

→ Differences from GRAND mean

→ INTER-Individual Differences

Within-Person Variance:

→ Differences from OWN mean

→ INTRA-Individual Differences

→ *This part is only observable through longitudinal data*

General Linear Model for +Within-Person Analysis

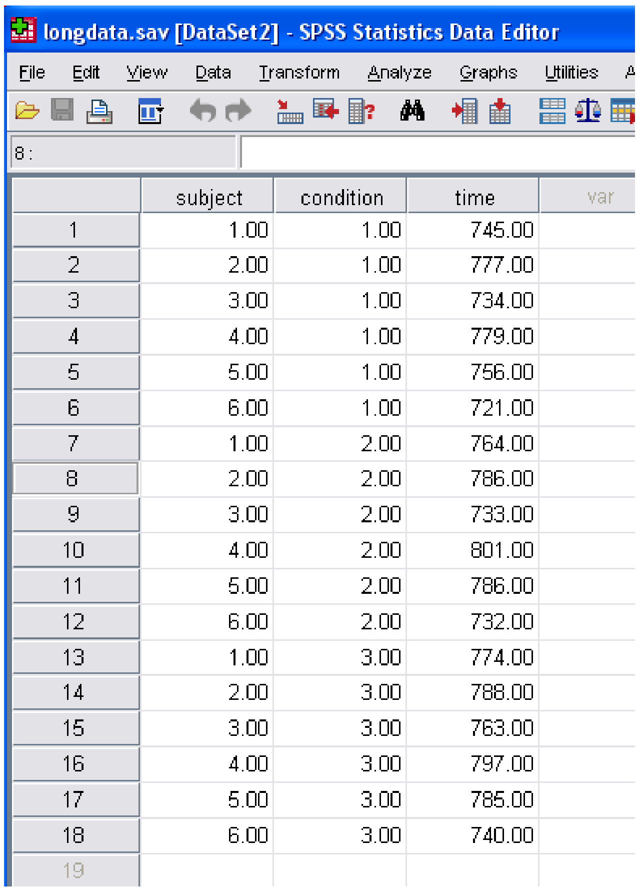
$$Y_{ij} = \boxed{\mu_T + \alpha_j} + \boxed{S_i + E_{ij}}$$

- **Model for the Means (Predicted Values):**
  - Same model for means, except that **DV and IV are measured more than once per person** (predicted Y per treatment/time per person)

- **Model for the Variance (2 piles now):**
  - $E_{ij} \sim N(0, \sigma_e^2) \rightarrow e_{ti}$ has a mean of 0 and some estimated constant variance (σ_e^2), is normally distributed, is unrelated to the IV, and is **unrelated across people and time**
  - **$S_i \sim N(0, \tau^2) \rightarrow$ mean differences across people**
  - **S_i has a mean of 0 and some estimated constant variance (τ^2), is normally distributed, is unrelated to the IV, and is unrelated across people (constant over time within a person)**

Mixed Models in SPSS

- Long Data Needed:



	subject	condition	time	var
1	1.00	1.00	745.00	
2	2.00	1.00	777.00	
3	3.00	1.00	734.00	
4	4.00	1.00	779.00	
5	5.00	1.00	756.00	
6	6.00	1.00	721.00	
7	1.00	2.00	764.00	
8	2.00	2.00	786.00	
9	3.00	2.00	733.00	
10	4.00	2.00	801.00	
11	5.00	2.00	786.00	
12	6.00	2.00	732.00	
13	1.00	3.00	774.00	
14	2.00	3.00	788.00	
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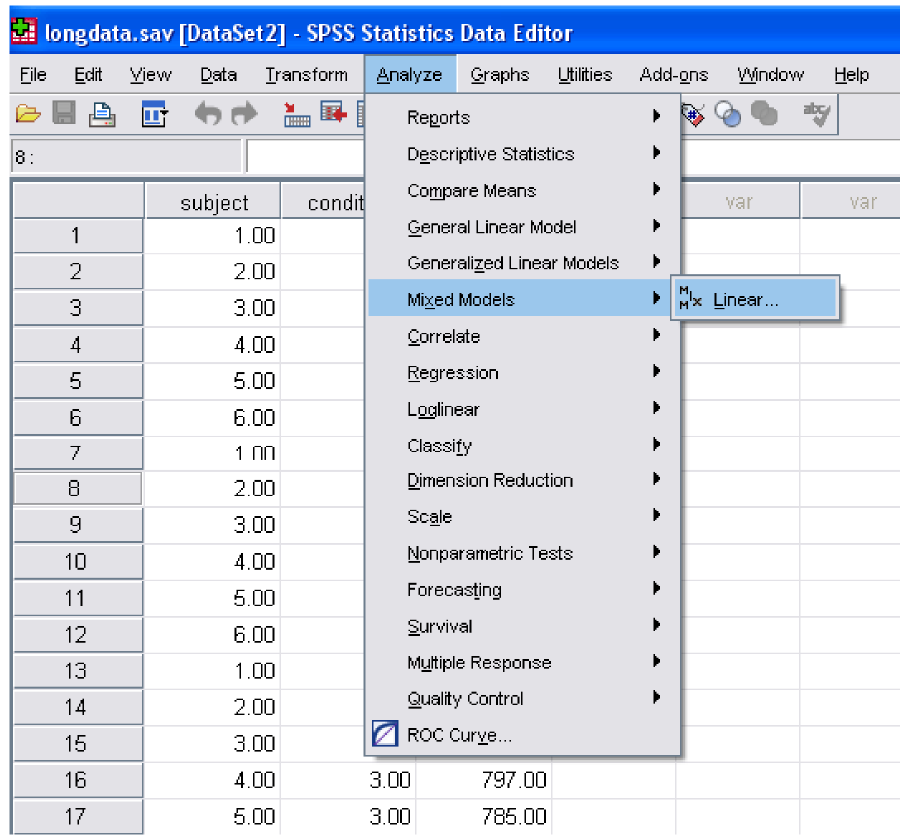
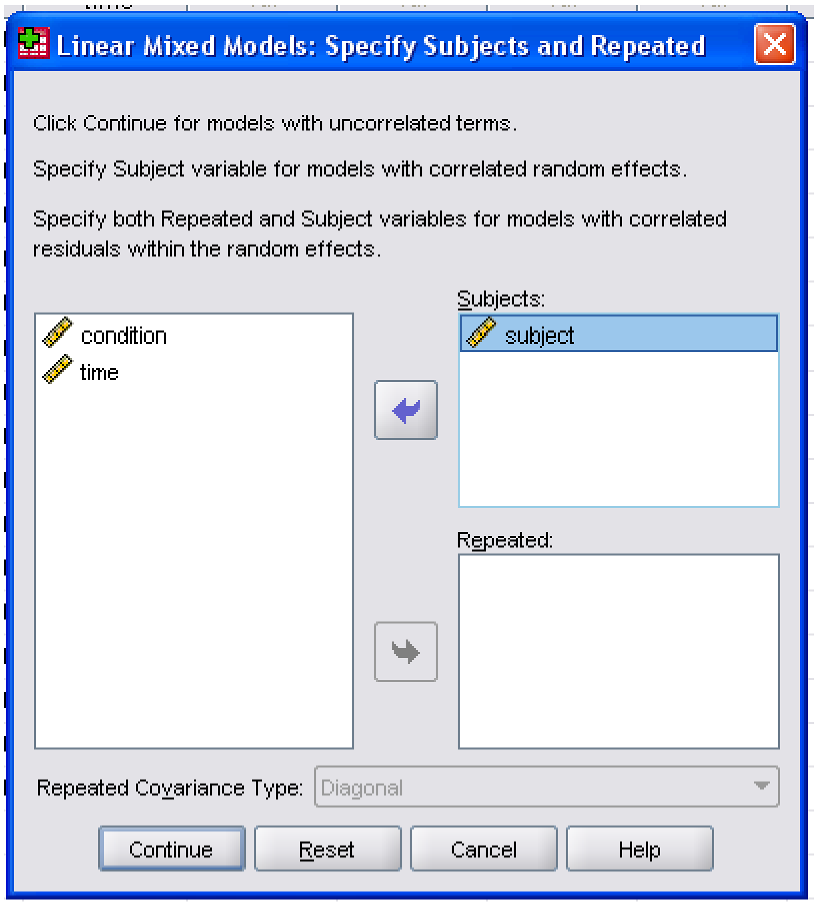
19				

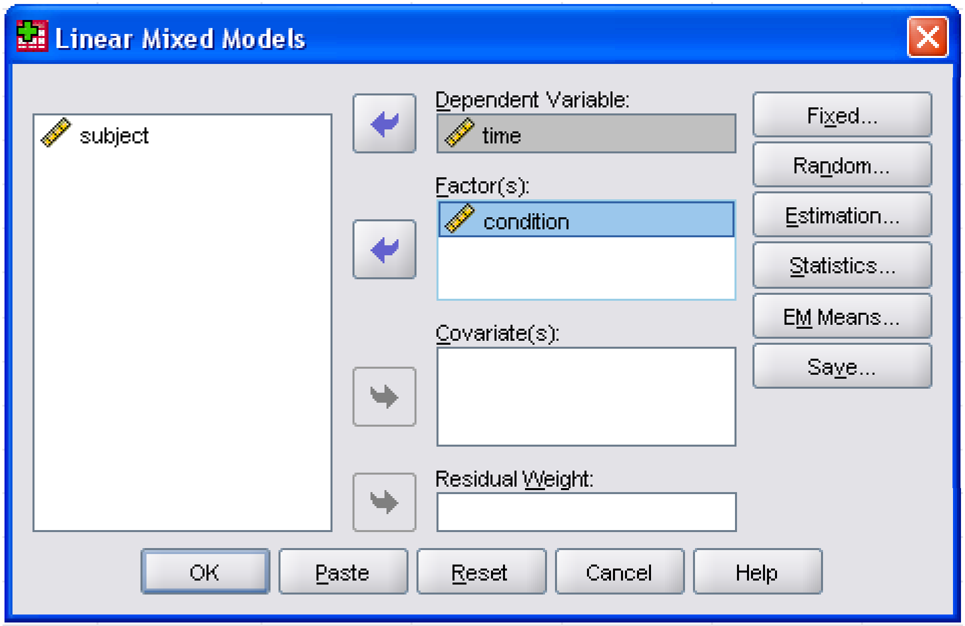
Mixed Models:



Start By Putting Subject Variable in…



Then Add DV and Factors…And Choose Fixed/Random



Three Analyses

- We will run all 3 types of ANOVA models from here:
  - Independence
  - Compound Symmetry (Now Random Intercept…was Repeated Measures)
  - Unstructured (was Multivariate)

- We will see the same results, but will get a sense of what covariance structure to use
  - Can use most powerful test for fixed effects

Independence ANOVA, Old and New

- Numbers from old:

$$F_A = 1.299; p = 0.302$$

$$\hat{\sigma}^2_{error} = 606.233$$

$$\hat{\mathbf{V}} = \begin{bmatrix} 606.233 & 0 & 0 \\ 0 & 606.233 & 0 \\ 0 & 0 & 606.233 \end{bmatrix}$$

Information Criteria[a]

-2 Restricted Log Likelihood	144.052
Akaike's Information Criterion (AIC)	146.052
Hurvich and Tsai's Criterion (AICC)	146.360
Bozdogan's Criterion (CAIC)	147.760
Schwarz's Bayesian Criterion (BIC)	146.760

The information criteria are displayed in smaller-is-better forms.

a. Dependent Variable: time.

Fixed Effects

Type III Tests of Fixed Effects[a]

Source	Numerator df	Denominator df	F	Sig.
Intercept	1	15	17353.524	.000
condition	2	15	1.299	.302

a. Dependent Variable: time.

Covariance Parameters

Estimates of Covariance Parameters[a]

Parameter	Estimate	Std. Error
Residual	606.233333	221.365115

a. Dependent Variable: time.

Compound Symmetry/Random Intercept

$$F_A = 14.432; p = 0.001$$

$$\hat{\sigma}^2_{error} = \hat{\sigma}^2_{AxS} = 54.567$$

$$\hat{\mathbf{V}} = \begin{bmatrix} 606.2 & 551.7 & 551.7 \\ 551.7 & 606.2 & 551.7 \\ 551.7 & 551.7 & 606.2 \end{bmatrix}$$

Information Criteria[a]

-2 Restricted Log Likelihood	125.158
Akaike's Information Criterion (AIC)	129.158
Hurvich and Tsai's Criterion (AICC)	130.158
Bozdogan's Criterion (CAIC)	132.574
Schwarz's Bayesian Criterion (BIC)	130.574

The information criteria are displayed in smaller-is-better forms.

a. Dependent Variable: time.

Fixed Effects

Type III Tests of Fixed Effects[a]

Source	Numerator df	Denominator df	F	Sig.
Intercept	1	5.000	6153.773	.000
condition	2	10.000	14.432	.001

a. Dependent Variable: time.

Covariance Parameters

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error
Residual		54.566667	24.402955
Intercept [subject = subject]	Variance	551.666667	360.500082

a. Dependent Variable: time.

Unstructured

- Multivariate tests…not run

$$\hat{\mathbf{V}} = \begin{bmatrix} 540.8 & 637.4 & 453.4 \\ 637.4 & 853.6 & 564.2 \\ 453.4 & 564.2 & 424.3 \end{bmatrix}$$

Information Criteria[a]

-2 Restricted Log Likelihood	120.397
Akaike's Information Criterion (AIC)	132.397
Hurvich and Tsai's Criterion (AICC)	142.897
Bozdogan's Criterion (CAIC)	142.645
Schwarz's Bayesian Criterion (BIC)	136.645

The information criteria are displayed in smaller-is-better forms.

a. Dependent Variable: time.

Fixed Effects

Type III Tests of Fixed Effects[a]

Source	Numerator df	Denominator df	F	Sig.
Intercept	1	5.000	6153.773	.000
condition	2	5.000	28.392	.002

a. Dependent Variable: time.

Covariance Parameters

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error
Repeated Measures	UN (1,1)	540.800000	342.031952
	UN (2,1)	637.400000	416.630686
	UN (2,2)	853.600000	539.864042
	UN (3,1)	453.400000	294.968812
	UN (3,2)	564.200000	368.918452
	UN (3,3)	424.300000	268.350882

a. Dependent Variable: time.

What One is Best? Use Information Criteria

- We can use the model with the smallest information criterion
  - For simplicity we will use BIC

- Independence: 146.760
- Compound Symmetry: 130.574
- Unstructured: 136.645

- The winner is:
  - Compound symmetry – our RM ANOVA model
  - We can now interpret the findings

Wrapping Up…

- Mixed models are powerful tools that are frequently used in research designs
  - Repeated measures
  - Longitudinal data
  - Hierarchical data
- Today’s class was a first pass at how the models work and what comes from them
  - To learn more, take a course in HLM or growth models
- Thank you for a great semester

Up Next…

- Final discussion

- In Lab:
  - How to do mixed models in SPSS

- Homework:
  - Study for the final (12/16 at 3:30 pm)

- Next week:
  - No class (reading day)
    - Have a good break

- The week after:
  - Final exam here at 3:30pm on 12/16