The University of Georgia

Analytical Comparisons Among Treatment Means (Chapter 4) Analysis of Trend (Chapter 5)

ERSH 8310
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Today's Class

- Chapter 4 – Analytic comparisons
  - The need for analytic comparisons
  - Planned comparisons
  - Comparisons among treatment means
  - Evaluating contrasts with t-tests
  - Orthogonal contrasts

- Chapter 5 - Analysis of trends
  - Using contrasts to do something a bit more practical
    - Linear trends
    - Quadratic trends

Still Vigilant…

TODAY'S EXAMPLE DATA SET

Vigilance Task While Sleep Deprived

- There are a = 4 conditions, namely, 4, 12, 20, and 28 hours without sleep
- There are n = 4 subjects randomly assigned to each of the different treatment conditions
- The vigilance task score represents the number of failures to spot objects on a radar screen during a 30-minute test period

Data (computation p. 51)

Hours without sleep			
4 hr	12 hr	20 hr	28 hr
a_1	a_2	a_3	a_4
37	36	43	76
22	45	75	66
22	47	66	43
25	23	46	62
Mean: 26.50	Mean: 37.75	Mean: 57.50	Mean: 61.75

SPSS Results

Descriptives

Errors

	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean Lower Bound	95% Confidence Interval for Mean Upper Bound	Minimum	Maximum
1.00	4	26.5000	7.14143	3.57071	15.1364	37.8636	22.00	37.00
2.00	4	37.7500	10.93542	5.46771	20.3493	55.1507	23.00	47.00
3.00	4	57.5000	15.50269	7.75134	32.8318	82.1682	43.00	75.00
4.00	4	61.7500	13.81726	6.90863	39.7637	83.7363	43.00	76.00
Total	16	45.8750	18.47476	4.61869	36.0305	55.7195	22.00	76.00

ANOVA

Errors

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	3314.250	3	1104.750	7.343	.005
Within Groups	1805.500	12	150.458		
Total	5119.750	15			

THE NEED FOR ANALYTIC COMPARISONS

The Need for Analytic Comparisons

- This chapter focuses on the analysis of experiments in which the independent variable consists of qualitative differences among the treatment conditions
  - The interest is in comparing specific treatment conditions

- An analytical comparison:
  - A meaningful comparison between two or more treatment conditions that are components of a larger experimental design (i.e., planned or post hoc comparisons)

The Composite Nature of SS_A

- For an equal n observations per group:

$$SS_A = n\sum_{j=1}^{a}\left(\overline{Y}_j - \overline{Y}_T\right)^2$$

The Composite Nature of SS_A

- Because:

$$\bar{Y}_T = \frac{\sum_{j=1}^{a} \bar{Y}_j}{a}$$

- We can re-express SS_A as:

$$SS_A = \frac{n}{a} \sum_{pairs} \left(\bar{Y}_i - \bar{Y}_k \right)^2$$

Example Decomposition

	A	B	C	D
1	Comparison	Mean i	Mean j	(M_i-M_j)^2
2	1 v. 2	26.5	37.75	126.5625
3	1 v. 3	26.5	57.5	961
4	1 v. 4	26.5	61.75	1242.5625
5	2 v. 3	37.75	57.5	390.0625
6	2 v. 4	37.75	61.75	576
7	3 v. 4	57.5	61.75	18.0625
8	n/a = 1			
9	Sum*(n/a)			3314.25

The Omnibus F Test

- The overall variation among the treatment means reflected in SS_A may be better understood by examining these contributing parts
  - The comparisons between pairs of means

- An F ratio based on more than two treatment levels is called the omnibus or overall F test

- Identifying the sources that contribute to the significant overall F should be performed to understand differences among the treatment means

PLANNED COMPARISONS

Planned Comparisons

- Analytical comparisons conducted directly on a set of data without reference to the result of the omnibus F test is possible

- These are called the planned comparisons

Planned Comparisons

- For instance, instead of testing the overall hypothesis
  - H_0: $\mu_1 = \mu_2 = \mu_3 = \mu_4$

- We may perform the following four comparisons:

$$
\begin{array}{ll}
(1) & H_0 : \mu_1 = \mu_2 \\
(2) & H_0 : \mu_1 = \mu_4 \\
(3) & H_0 : \mu_2 = \mu_3 \\
(4) & H_0 : \mu_3 = \mu_4
\end{array}
$$

COMPARISONS AMONG TREATMENT MEANS

Comparisons Among Treatment Means

- Comparisons that can be reduced to miniature experiment are called also contrasts or single-df comparisons
  - For each A = 2 (i.e., df = 1)

- We may extend the comparisons to situations where the subsets of three or more means are tested.

Linear Contrasts

- From the null hypothesis of a comparison we may represent the difference between two means with ψ

For example: $H_0 : \mu_1 = \dfrac{\mu_2 + \mu_3}{2}$

Can be rewritten: $H_0 : \mu_1 - \left(\dfrac{\mu_2 + \mu_3}{2} \right) = 0 \longrightarrow \psi = 0$

Yielding: $\psi = (+1)(\mu_1) + \left(-\dfrac{1}{2} \right)(\mu_2) + \left(-\dfrac{1}{2} \right)(\mu_3)$

Linear Contrasts

- From the null hypothesis of a comparison we may represent the difference between two means with ψ

For example:

$$H_0 : \mu_1 = \frac{\mu_2 + \mu_3 + \mu_4}{3}$$

Can be rewritten:

$$H_0 : \mu_1 - \left(\frac{\mu_2 + \mu_3 + \mu_4}{3} \right) = 0 \longrightarrow \psi = 0$$

Yielding:

$$\psi = (+1)(\mu_1) + \left(-\frac{1}{3}\right)(\mu_2) + \left(-\frac{1}{3}\right)(\mu_3) + \left(-\frac{1}{3}\right)(\mu_4)$$

The New Null Hypothesis

- From the last slide, our contrast is:

$$\psi = (+1)(\mu_1) + \left(-\frac{1}{2}\right)(\mu_2) + \left(-\frac{1}{2}\right)(\mu_3)$$

- The numbers multiplied by each mean are called coefficients
  - Here they are +1, -.5, and -.5

- Note that, now, H_0: $\psi = 0$
  - Our coefficients sum to zero

Contrasts, In General

- The general formula is:

$$\psi = \sum_{j=1}^{a} c_j \mu_j$$

- Here, c_j represents the coefficient multiplying group j
- There is a constraint that the coefficients sum to zero
  - Keeps the null hypothesis such that $\psi = 0$

Vigilance Example Contrasts

- For our vigilance example, the contrasts would be:

a_1	a_2	a_3	a_4
Mean: 26.50	Mean: 37.75	Mean: 57.50	Mean: 61.75

$$(1) \quad H_0 : \mu_1 = \mu_2 \rightarrow \hat{\psi} = \overline{Y}_1 - \overline{Y}_2 = 26.50 - 37.75 = -11.25$$

$$(2) \quad H_0 : \mu_1 = \mu_4 \rightarrow \hat{\psi} = \overline{Y}_1 - \overline{Y}_4 = 26.50 - 61.75 = -35.25$$

$$(3) \quad H_0 : \mu_2 = \mu_3 \rightarrow \hat{\psi} = \overline{Y}_2 - \overline{Y}_3 = 37.75 - 57.50 = -19.75$$

$$(4) \quad H_0 : \mu_3 = \mu_4 \rightarrow \hat{\psi} = \overline{Y}_3 - \overline{Y}_4 = 57.50 - 61.75 = -4.25$$

Constructing Coefficients

- *Pairwise comparison*: a comparison between two groups (e.g., $\psi = \mu_j - \mu_k$)

- *Complex comparison*: a comparison between an average of two or more groups and either a single group or an average of two or more groups

$$\psi = \frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4}{2}$$

- If the coefficients are expressed as a set of numbers ({.5, .5, -.5, -.5}), we call them the relative weights of the groups

The Sums of Squares for a Comparison

- If

$$\hat{\psi} = \sum_{j=1}^{a} \mathrm{c}_j \overline{\mathrm{Y}}_j$$

- then the sum of squares for the contrast is:

$$\mathrm{SS}_{\psi} = \frac{\mathrm{n}\hat{\psi}^2}{\sum_{j=1}^{a} \mathrm{c}_j^2}$$

Vigilance Example

- For the vigilance example:

$$(1) \quad \hat{\psi}_1 = -11.25 \rightarrow SS_{\psi_1} = \frac{4 * -11.25^2}{1^2 + (-1)^2 + 0 + 0} = 253.125$$

$$(2) \quad \hat{\psi}_2 = -35.25 \rightarrow SS_{\psi_2} = \frac{4 * -35.25^2}{1^2 + 0 + 0 + (-1)^2} = 2485.125$$

$$(3) \quad \hat{\psi}_3 = -19.75 \rightarrow SS_{\psi_3} = \frac{4 * -19.75^2}{0 + 1^2 + (-1)^2 + 0} = 780.125$$

$$(4) \quad \hat{\psi}_4 = -4.25 \rightarrow SS_{\psi_4} = \frac{4 * -4.25^2}{0 + 0 + 1^2 + (-1)^2} = 36.125$$

Evaluating Comparisons

- If the comparisons are planned comparisons, we may not even bother to perform the omnibus F test

- For each contrast, however, the F ratio is formed as

$$F = \frac{MS_{\psi}}{MS_{S/A}}$$

where $df_A = 1$ and $df_{S/A} = A(n-1)$

- Our null hypothesis is: $H_{0:}\ \Psi = 0$
- Our alternative hypothesis is: $H_{1:}\ \Psi \neq 0$

Vigilance, Continued

- From our vigilance example:
- $MS_{S/A} = 150.458$
- $df_{\psi} = 1; df_{S/A} = 12$

ANOVA

Errors

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	3314.250	3	1104.750	7.343	.005
Within Groups	1805.500	12	150.458		
Total	5119.750	15			

$$(1)\quad F_{\hat{\psi}_1} = \frac{SS_{\hat{\psi}_1}/df_{\psi}}{MS_{S/A}} = \frac{253.125}{150.458} = 1.68; p = 0.219$$

$$(2)\quad F_{\hat{\psi}_2} = \frac{SS_{\hat{\psi}_2}/df_{\psi}}{MS_{S/A}} = \frac{2485.125}{150.458} = 16.52; p = 0.002$$

$$(3)\quad F_{\hat{\psi}_3} = \frac{SS_{\hat{\psi}_3}/df_{\psi}}{MS_{S/A}} = \frac{780.125}{150.458} = 5.19; p = 0.042$$

$$(4)\quad F_{\hat{\psi}_4} = \frac{SS_{\hat{\psi}_4}/df_{\psi}}{MS_{S/A}} = \frac{36.125}{150.458} = 0.240; p = 0.633$$

P-values obtained using “fdist” function in Excel

Unequal Sample Sizes

- Up to this point, our formulae were for equal sample sizes within group
  - Not likely in practice
- Let n_j indicate the number of subjects in group a_j.
- The contrast sums of squares is

$$SS_{\psi} = \frac{\hat{\psi}^2}{\sum_{j=1}^{a} \frac{c_j^2}{n_j}}$$

- MS_{ψ} and F can be obtained accordingly

ORTHOGONAL CONTRASTS

Orthogonal Contrasts

- We can divide any sum of squares into as many independent sums of squares as there are df

- Orthogonal comparisons reflect nonoverlapping pieces of information

- The outcome of one comparison gives no indication about the outcome of any other orthogonal comparison

- Orthogonality means independence of information

What is Orthogonal?

- A numerical test of the orthogonality of any two comparisons is provided by the following relationship between the two sets of coefficients:

$$\sum_{j=1}^{a} c_{1j} c_{2j} = 0$$

c_{1j} and c_{2j} are corresponding coefficients in the two comparisons.

Are We Orthogonal?

- Our contrasts and coefficients:

Contrast	c_{j1}	c_{j2}	c_{j3}	c_{j4}
(1)	1	-1	0	0
(2)	1	0	0	-1
(3)	0	1	-1	0
(4)	0	0	1	-1

- Sum of crossproducts ($\sum_{j=1}^{a} c_{1j} c_{2j}$):

Contrast	(1)	(2)	(3)	(4)
(1)				
(2)	1			
(3)	-1	0		
(4)	0	1	-1	

Orthogonality and ANOVA

- The number of orthogonal comparisons for a given set of data is a-1,
  - The df for the SS_A (e.g., Helmert contrasts)
- No in agreement on the issue of placing orthogonality restrictions on the nature of the planned comparisons
- "Researchers must exercise judgment in the planning stages to guarantee that the important questions studies in an investigation can be answered unambiguously by the proposed experimental design"
- Nonorthogonal comparisons require special care to avoid logical ambiguities

COMPOSITE CONTRASTS DERIVED FROM THEORY

Contrast Coefficients that Match a Pattern

- The contrast coefficients that reflect theoretical pattern can be constructed using the steps:
  1. Use each predicted mean as a starting coefficient
  2. Subtract the average of these means from the predicted means so that they sum to zero
  3. Optionally, simplify the coefficients to be integers

- The testing of the observed pattern of means with the theoretically-derived expectation as well as the assessment of the fit of the outcome to the predicted pattern can be performed

Chapter 5

Analysis of Trend

(or – specific types of contrasts)

ANALYSIS OF LINEAR TREND

Trend Analysis

- Trend analysis is a specialized form of single-df comparisons when a quantitative independent variable is manipulated

- Treatment levels represent different amounts of a single common variable
  - As in our vigilance task

- We usually plot the entire set of treatment means on a graph, connect the points, and examine the display for any underling shape or trend

Data (computation p. 51)

Hours without sleep			
4 hr	12 hr	20 hr	28 hr
a_1	a_2	a_3	A_4
37	36	43	76
22	45	75	66
22	47	66	43
25	23	46	62
Mean: 26.50	Mean: 37.75	Mean: 57.50	Mean: 61.75

Testing for Linear Trend

- A way of assessing linear trend is to use a set of ***coefficients*** that represents an idealized version of a straight line (see Appendix A.3 on p. 577)

- There are A-1 ***orthogonal*** polynomials possible

- For our example, the linear trend would be:

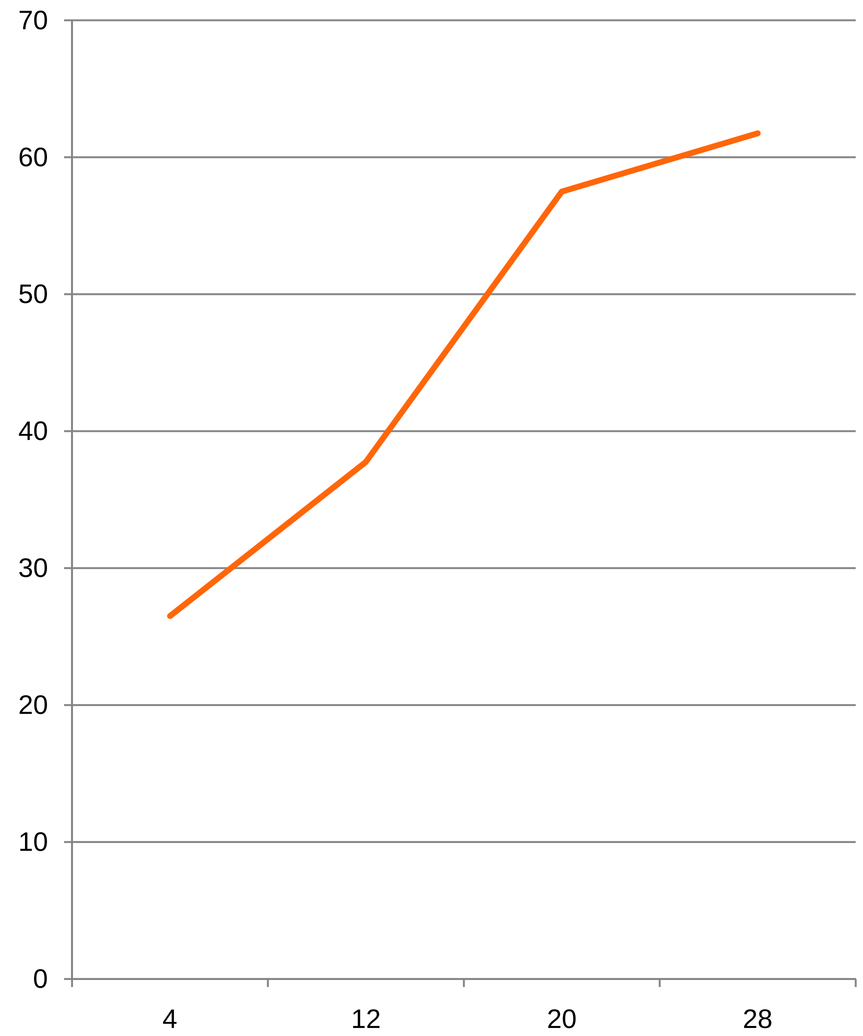
c_1	c_2	c_3	c_4
-3	-1	1	3

The Single-df Linear Trend Test

- The tests used in Chapter 4 are all the same
  - Talk about convenient!
- First, form the contrast: $$\hat{\psi} = \sum_{j=1}^{a} c_j \overline{Y}_j$$
- Then, compute the SS: $$SS_{\psi} = \frac{n\hat{\psi}^2}{\sum_{j=1}^{a} c_j^2}$$
- Then, construct the F: $$F = \frac{MS_{\psi}}{MS_{S/A}}$$

Mean Plot

- One of the easiest visual plots possible is to look at the mean for each group
- X-axis: IV level
- Y-axis: Group mean
- Does this look linear?



Linear Trend in Vigilance?

- We will now test for a linear trend in vigilance
- 1. We construct our contrast:

$$\hat{\psi} = \sum_{j=1}^{a} c_j \overline{Y}_j = -3 * 26.5 + -1 * 37.75 + 1 * 57.5 + 3 * 61.75$$

$$\hat{\psi} = 125.5$$

- 2. We construct the contrast sums of squares

$$SS_{\psi} = \frac{n\hat{\psi}^2}{\sum_{j=1}^{a} c_j^2} = \frac{4 * 125.5^2}{(-3)^2 + (-1)^2 + 1^2 + 3^2} = 3150.05$$

Linear Trend in Vigilance?

- 3. We compute the contrast F-statistic

$$F = \frac{MS_{\psi}}{MS_{S/A}} = \frac{3150.05}{150.458} = 20.94$$

ANOVA

Errors

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	3314.250	3	1104.750	7.343	.005
Within Groups	1805.500	12	150.458		
Total	5119.750	15			

- 4. We compute the p-value (fdist(20.94,1,12)):
  - $p = 0.001$

- From this, we ***reject*** the null hypothesis
  - There is a linear trend in vigilance scores across sleep deprivation

ANALYSIS OF QUADRATIC TREND

Analysis of Quadratic Trend

- A quadratic trend is one that displays concavity, a single bend either upward or downward
- The coefficients of orthogonal polynomials in Appendix A.3 can be used to test quadratic trend
- The testing of quadratic trend can be accomplished by the formulae on the next slide

The Single-df Quadratic Trend Test

- The tests used in Chapter 4 are STILL all the same
  - Talk about convenient!

- First, form the contrast: $$\hat{\psi} = \sum_{j=1}^{a} c_j \overline{Y}_j$$

- Then, compute the SS: $$SS_{\psi} = \frac{n\hat{\psi}^2}{\sum_{j=1}^{a} c_j^2}$$

- Then, construct the F: $$F = \frac{MS_{\psi}}{MS_{S/A}}$$

Our Example

- Quadratic contrast coefficients (Appendix A.3):

c_1	c_2	c_3	c_4
1	-1	-1	1

- 1. We construct our contrast:

$$\hat{\psi} = \sum_{j=1}^{a} c_j \overline{Y}_j = 1*26.5 + -1*37.75 + -1*57.5 + 1*61.75$$

$$\hat{\psi} = -7.00$$

- 2. We construct the contrast sums of squares

$$SS_{\psi} = \frac{n\hat{\psi}^2}{\sum_{j=1}^{a} c_j^2} = \frac{4*-7^2}{1^2 + (-1)^2 + (-1)^2 + 1^2} = 49$$

Quadratic Trend in Vigilance?

- 3. We compute the contrast F-statistic

$$F = \frac{MS_{\psi}}{MS_{S/A}} = \frac{49}{150.458} = 0.33$$

ANOVA

Errors

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	3314.250	3	1104.750	7.343	.005
Within Groups	1805.500	12	150.458		
Total	5119.750	15			

- 4. We compute the p-value (fdist(0.33,1,12)):
  - p = 0.579

- From this, we ***retain*** the null hypothesis
  - There is *not* a quadratic trend in vigilance scores across sleep deprivation

Higher-Order Trend Components

- Testing for Higher-Order Trends:
  - A curve that has two reversals is called a cubic trend component, and one that has three reversals is called a quartic trend component
  - The higher-order trend can be tested with the coefficients in Appendix A.3.
  - For example, we may calculate the cubic trend for a = 4

- In general, you keep testing until you retain a null hypothesis for a trend
  - We retained the quadratic so we could have stopped
  - Will continue to demonstrate contrast effects

Our Example

- The cubic contrast coefficients (Appendix A.3):

c_1	c_2	c_3	c_4
-1	3	-3	1

- 1. We construct our contrast:

$$\hat{\psi} = \sum_{j=1}^{a} c_j \overline{Y}_j = -1*26.5 + 3*37.75 + -3*57.5 + 1*61.75$$

$$\hat{\psi} = -24.00$$

- 2. We construct the contrast sums of squares

$$SS_{\psi} = \frac{n\hat{\psi}^2}{\sum_{j=1}^{a} c_j^2} = \frac{4*-24}{(-1)^2 + 3^2 + (-3)^2 + 1^2} = 115.2$$

Cubic Trend in Vigilance?

- 3. We compute the contrast F-statistic

$$F = \frac{MS_{\psi}}{MS_{S/A}} = \frac{115.2}{150.458} = 0.77$$

ANOVA

Errors

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	3314.250	3	1104.750	7.343	.005
Within Groups	1805.500	12	150.458		
Total	5119.750	15			

- 4. We compute the p-value (fdist(0.766,1,12)):
  - $p = 0.399$

- From this, we ***retain*** the null hypothesis
  - There is *not* a cubic trend in vigilance scores across sleep deprivation

Are We Orthogonal?

- The three previous contrasts were orthogonal:

Contrast	c_{j1}	c_{j2}	c_{j3}	c_{j4}
(1)	-3	-1	1	3
(2)	1	-1	-1	1
(3)	-1	3	-3	1

- Sum of crossproducts ($\sum_{j=1}^{a} c_{1j}c_{2j}$):

Contrast	(1)	(2)	(3)
(1)			
(2)	0		
(3)	0	0	

Contrast Orthogonality

- When contrasts are orthogonal and complete (using all A-1 of them), the sum of SS_{ψ} sums to SS_A
- $SS_{linear} = 3150.05$
- $SS_{quadratic} = 49$
- $SS_{cubic} = 115.2$
- $SS_{linear} + SS_{quadratic} + SS_{cubic} = 3314.25$

ANOVA

Errors

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	3314.250	3	1104.750	7.343	.005
Within Groups	1805.500	12	150.458		
Total	5119.750	15			

PLANNING A TREND ANALYSIS

Issues in Trend Analysis

- There are two questions that surround the choice of intervals:
  - The nature of the spacing between adjacent levels
  - The number of intervals

- It is better to use equally spaced intervals

- Trend Coefficients for Unequal Intervals:
  - Coefficients can be found by simply picking numbers that agree with the actual spacing and adjust them to sum to zero
- The Number of Intervals:
  - Seriously consider investing in a substantial experiment that includes a sufficient number of treatment conditions to provide convincing evidence of the trends

Other Issues

- Other Functional Forms
  - Other functions (e.g., exponential or logarithm) can be used

- Monotonic Trend Analysis
  - When we have only predicted the rank order of the conditions, we have a monotonic hypothesis based on a monotonic function

  - We may apply the logic behind the trend coefficients in Appendix A.3 appropriately modified for a given problem

Wrapping Up…

- Contrasts are specific hypothesis tests that examine how each mean may differ from all the other means

- Trend analysis takes the idea of contrasts and maps it onto looking at trends

- Trend analysis is typically conducted when the factor levels have some understandable scale

- All contrasts are executed the same way – convenient!

Up Next…

- In Lab:
  - How to do contrasts and tests for trends in SPSS
- Homework
  - Up on ELC now, due at the start of class on 9/16
- Reading for next week:
  - Chapter 6: Pairwise Comparisons