

SOURCES OF VARIABILITY AND SUMS OF SQUARES CHAPTER 2

Today's Class

- The logic of hypothesis testing.
- Component deviations.
 - ▣ Sums of squares.
- Computational formulas for sums of squares.



The Logic of Hypothesis Testing

Differences Between Groups

- Differences observed among treatment means are influenced jointly by the actual differences in the treatments administered to the different groups and by chance factors introduced by randomization.
- The decision confronting the experimenter is to decide whether the differences associated with the treatment conditions are entirely or just partly due to chance.

The Logic of Hypothesis Testing

- Our main goal is to make inferences about the behavior of subjects who have not been tested in our experiment (i.e., population) on the basis of the experimental outcome from the subjects in the experiment (i.e., sample).
- Summary descriptions calculated from the data of a sample are called statistics, and measures calculated from all the observations with the population are called parameters.
- In most cases, Roman letters designate statistics and Greek letters designate parameters.

Statistical Hypotheses

- Typically, a research hypothesis asserts that the treatments will produce an effect.
- Statistical hypotheses consist of a set of precise hypotheses about the parameters of the different treatment populations.

Null Hypothesis

- There are two statistical hypotheses, the null and the alternative.
- The statistical hypothesis to be tested is called the null hypothesis H_0 .
- For example, the null hypothesis of the analysis of variance with three treatment conditions can be:

$$H_0: \mu_1 = \mu_2 = \mu_3.$$

Alternative Hypothesis

- The alternative hypothesis specifies the values for the parameter that are incompatible with the null hypothesis. For example,

H_1 : not all μ 's are equal.

Experimental Error

- All nuisance variables that we control in our experiment through random assignment of subjects to the treatment conditions are considered potential contributors to experimental error.
- In the behavioral sciences, the most important source of experimental error is that due to individual differences.
- Another source of experimental error is what may be called measurement error.
- We describe the contribution of all the different components of experimental error as unsystematic, stressing the fact that their influence is independent of the treatment effects.

Estimates of Experimental Error

- The variability of subjects treated alike, that is, within the same treatment level, provides an estimate of experimental error.
- If we assume that experimental error is the same for the different treatment conditions, we can obtain a more stable estimate of this quantity by pooling and averaging these separate estimates.
 - ▣ Assume that the null hypothesis is true.
 - ▣ The sample means nevertheless may be different.
 - ▣ If the sample means are not equal, the only reasonable explanation that we can offer for these differences is the operation of experimental error.

Estimates of Treatment Effects

- If the null hypothesis is false, there are real differences among the means of the treatment populations.
- A systematic component (as opposed to an unsystematic one) that contributes to the differences among the means is called the treatment effects.
 - ▣ Differences among treatment means may reflect two different quantities.
- When the population means are equal, the differences among the group means will reflect the operation of experimental error alone, but when the population means are not equal, the differences among the group means will reflect the operation of an unsystematic component (i.e., experimental error) and a systematic component (i.e., treatment effects).

Evaluation of the Null Hypothesis

- When the null hypothesis is true, two estimates of experimental error are available from the experiment.
- The ratio of the two estimates can produce a useful statistic.

$$\frac{\text{differences among treatment means}}{\text{differences among subjects treated alike}} = \frac{\text{experimental error}}{\text{experimental error}}$$

- We would expect to find an average value of this ratio of approximately 1.0.

Evaluation of the Null Hypothesis

- Consider now the same ratio when the null hypothesis is false. The ratio becomes:

$$\frac{\text{treatment error} + \text{experimental error}}{\text{experimental error}}$$

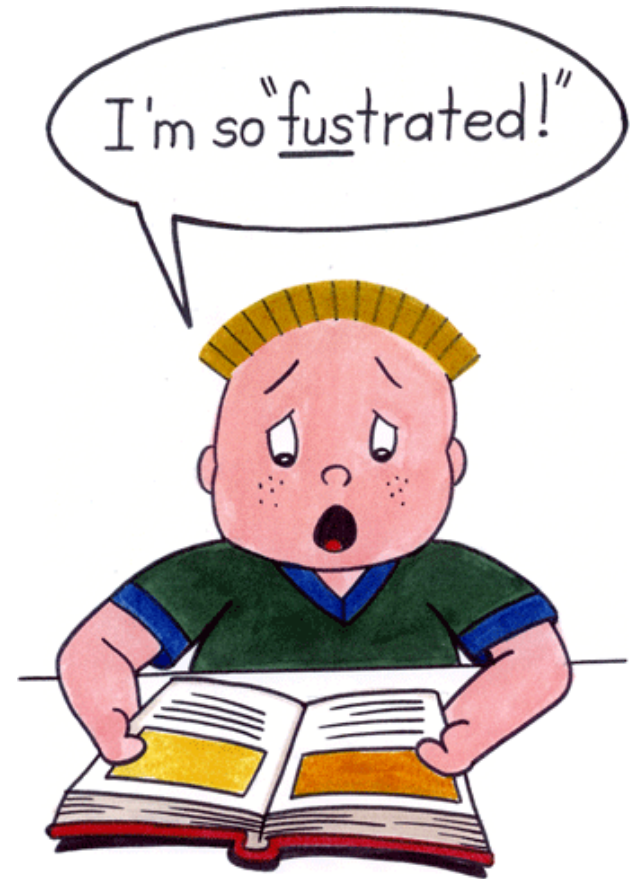
- If we were to repeat the experiment a large number of times, we would expect to find an average value of this ratio that is greater than 1.0.
- We will make a decision concerning the acceptability of the null hypothesis that is based on a consideration of the chance probability associated with the ratio we actually found in the experiment.
- If the probability of obtaining ratio of this size or larger by chance is reasonably low, we will reject the null hypothesis.



The Component Deviations

Component Deviations

- We will see the abstract notions of variability between treatment groups and within treatment groups become concrete arithmetic operations extracted from scores produced in single factor experiments.
- Suppose we were interested in the effect of therapeutic drugs on reading comprehension of hyperactive boys.



Example

- One group of boys serves as a control or placebo condition; a second group is given one of the drugs; and a third group is given the other.
- The independent variable, types of drugs is referred to as factor A.
- The three levels of factor A are denoted as a_1 , a_2 , and a_3 .
- The subjects are drawn from a fourth-grade class and randomly assigned to each of the levels of factor A.
- There are five subjects in each level.
- The response measure is the score for each subject, denoted by Y. It is the number of test items correctly answered by each student.

Example Data

Level α_1	Level α_2	Level α_3
16	4	2
18	7	10
10	8	9
12	10	13
19	1	11

$$\bar{Y}_{a_1} = 15$$

$$\bar{Y}_{a_2} = 6$$

$$\bar{Y}_{a_3} = 9$$

$$\bar{Y}_T = 10$$

The Grand Mean

Deviations

- Let us use the notation $Y_{i,j}$ to represent a score from the j^{th} level of factor A and the i^{th} subject based on the ordinal position within the level.
- Note we may also use Y_{ij} .
- For example, $Y_{5,2} = 1$.
- The deviation of $Y_{i,j}$ from the grand mean \bar{Y}_T can be seen as two components:

$$Y_{i,j} - \bar{Y}_T = (Y_{a_i} - \bar{Y}_T) + (Y_{i,j} - \bar{Y}_{a_i})$$

Deviations

- The total deviation is: $\left[\bar{y}_{i,j} - \bar{Y}_T \right]$
- The between groups deviation is: $\left[\bar{y}_{a_i} - \bar{Y}_T \right]$
- The within groups deviation is: $\left[\bar{y}_{i,j} - \bar{Y}_{a_i} \right]$

Deviations Explained

- The score for each subject in an experiment can be expressed as a deviation from the grand mean and that this deviation can be partitioned into two components, a between-groups deviation and a within-groups deviation.
- These two component deviations are what we have been after, a quantity that will reflect **treatment effects** in the population in addition to experimental error (i.e., the between-groups deviation), and a quantity that will reflect **experimental error** alone (i.e., the within-groups deviation).



Sums of Squares: Defining Formulas

Sums of Squares

- To evaluate the null hypothesis, it is necessary to transform the between-groups and within-groups deviations into more useful quantities, namely, variances.
- For this reason the statistical analysis involving the comparison of variances reflecting different sources of variability is called the analysis of variance.

Variance

- A variance is defined as follows:

$$\text{variance} = \frac{\text{sum of squared deviations from the mean} \quad SS}{\text{degrees of freedom} \quad \text{df}} = \frac{SS}{df}$$

- where df is approximately equal to the number of cases in the set.
- This means that the variance is basically an average of the squared deviations.

Handy Properties of Sums of Squares

- Note that:

$$SS_{\text{total}} = SS_{\text{between groups}} + SS_{\text{within groups}}$$

or, equivalently,

$$SS_T = SS_A + SS_{S/A}$$

- Where T denotes total.
- A denotes the factor.
- S/A denotes the subjects within the factor.

Total Sum of Squares

- The total sum of squares is formed by squaring the total deviation for each subjects and summing the squares of the total deviations:

$$SS_T = \sum (Y_{ij} - \bar{Y}_T)^2$$

- Where:

$$\bar{Y}_T = \frac{\sum Y_{ij}}{N}$$

- In our example $SS_T = 390$.

Between-Groups Sums of Squares

- The between-groups sums of squares:

$$SS_A = \sum_{j=1}^a n_j (\bar{Y}_{A_j} - \bar{Y}_T)^2$$

Where a is the total number of groups or treatment levels and:

$$\bar{Y}_{A_j} = \sum_{i=1}^{n_j} \frac{Y_{ij}}{n_j}$$

Between Groups Sums of Squares

□ If $a=3$ then

$$SS_A = n_1 (\bar{Y}_{A_1} - \bar{Y}_T)^2 + n_2 (\bar{Y}_{A_2} - \bar{Y}_T)^2 + n_3 (\bar{Y}_{A_3} - \bar{Y}_T)^2$$

□ If all n are equal:

$$SS_A = n \sum_{j=1}^a (\bar{Y}_{A_j} - \bar{Y}_T)^2$$

□ In our example, $SS_A = 210$

Within-Groups Sums of Squares

- The within-groups sums of squares:

$$SS_{S/A} = \sum_{i=1}^a \sum_{j=1}^{n_i} (y_{ij} - \bar{Y}_{A_i})^2$$



Computational Formulas

Computational Formulas

- We usually calculate the sum of squares with formulas that are equivalent algebraically but much simpler computationally.
 - ▣ Because the book makes use of these formulae, we will highlight them here...
- In the analysis of the complete randomized single-factor design, we need to designate three basic quantities:
 - ▣ The individual scores or observations (the raw data).
 - ▣ The sum of these scores for each treatment condition (the treatment sums or subtotals).
 - ▣ The sum of all the scores or observations (the grand sum or grand total).

Individual Scores

- The individual scores are designated by Y .
- Sometimes Y with a subscript i or subscripts i and j can be used.
- For example, Y_{ij} or $Y_{i,j}$ can be used where the subscript j refers to the levels of the independent variable, factor A ($j = 1, \dots, a$), and the subscript i refers to the subject within the level j ($i = 1, \dots, n_j$).

Treatment Sums and Means

- The treatment sums, or subtotals, are

$$A_j = \sum_{i=1}^{n_j} Y_{ij}$$

- The treatment means are:

$$\bar{Y}_{A_j} = \frac{A_j}{n_j}$$

Grand Sum and Grand Mean

- The grand sum is the sum of all the scores in the experiment:

$$T = \sum_{i=1}^{n_j} \sum_{j=1}^a Y_{ij} = \sum_{j=1}^a A_j$$

- The grand mean is (with equal sample sizes):

$$\bar{Y}_T = \frac{T}{N} = \frac{T}{(a)(n)}$$

Basic Ratios

- Basic ratios represent a common step in the computational formulas for sums of squares in the analysis of variance.

- There are three basic ratios:

- ▣ One involving the individual observation Y .

$$Y_{..}^2 = \sum_{i=1}^{n_j} \sum_{j=1}^a Y_{ij}^2$$

- ▣ Another involving the treatment subtotals A .

$$A_{..}^2 = \frac{\sum_{j=1}^a A_j^2}{n}$$

- ▣ The other involving the grand total T .

$$T^2 = \frac{T^2}{(a)(n)}$$

Sums of Squares

- The basic ratios are combined to produce the three required sums of squares:
- $SS_T = [Y] - [T]$
- $SS_A = [A] - [T]$
- $SS_{S/A} = [Y] - [A]$
- Also, note $SS_T = SS_A + SS_{S/A}$

Final Thought

- Sums of squares play an important role in determining if group differences observed in an experiment are due to chance or not.
- The formulae presented today provide the basics of how ANOVA works: by partitioning variance into distinct types:
 - Between groups
 - Within groups
- Such partitioning will be what is done throughout this class for nearly all techniques.



Next Class

- Chapter 3 of Keppel:
 - ▣ Variance Estimates and the F Ratio