Psychometric Models:
The Loglinear Cognitive Diagnosis Model

Section #3
NCME 2016 Training Session
Lecture Objectives

• Discuss relevant mathematical prerequisites for understanding diagnostic measurement models

• Introduce the loglinear cognitive diagnosis model – a general measurement model for DCMs

• Show some models the LCDM subsumes
Development of Psychometric Models

• Over the past several years, numerous DCMs have been developed
  ➢ We will focus on DCMs that use latent variables for attributes
  ➢ This lecture focuses on the only one you should pay attention to: the Loglinear Cognitive Diagnosis Model

• Each DCM makes assumptions about how mastered attributes combine/interact to produce an item response
  ➢ Compensatory/disjunctive/additive models
  ➢ Non-compensatory/conjunctive/non-additive models

• With so many models, analysts have been unsure which model would best fit their purpose
  ➢ Difficult to imagine all items following same assumptions
Recent developments have produced very general diagnostic models

- General Diagnostic Model (GDM; von Davier, 2005)
- Loglinear Cognitive Diagnosis Model (LCDM; Henson, Templin, & Willse, 2009)
  - Focus of this session
- Generalized DINA Model (G-DINA; de la Torre, 2011)
  - Is equivalent to the LCDM

The LCDM provides great modeling flexibility

- Subsume all other latent variable DCMs
- Allow both additive and non-additive relationships between attributes/items
- Sync with other psychometric models allowing for greater understanding of modeling process
Lecture Overview

• Background information
  • ANOVA models and the LCDM

• Logits explained

• The LCDM
  • Parameter structure
  • One-item demonstration

• LCDM general form

• Linking the LCDM to other earlier-developed DCMs
Notation Used Throughout Session

- **Attributes**: \( a = 1, \ldots, A \)

- **Respondents**: \( r = 1, \ldots, R \)

- **Attribute Profiles**: \( \alpha_r = [\alpha_{r1}, \alpha_{r2}, \ldots, \alpha_{rA}] \)
  - Each attribute \( \alpha_{ra} \) today is defined as being 0 or 1: \( \alpha_{ra} \in \{0,1\} \)

- **Latent Classes**: \( c = 1, \ldots, C \)
  - We have \( C = 2^A \) latent classes – one for each possible attribute profile
  - An attribute profile is a specific permutation of all \( A \) attributes

- **Items**: \( i = 1, \ldots, I \)
  - Restricted to dichotomous item responses (either 0 or 1): \( Y_{ri} \in \{0,1\} \)

- **Q-matrix**: Elements \( q_{ia} \) are indicators an item \( i \) measures attribute \( a \)
  - \( q_{ia} \) is either 0 (does not measure \( a \)) or 1 (measures \( a \)): \( q_{ia} \in \{0,1\} \)
BACKGROUND INFORMATION: ANOVA MODELS
• The LCDM models the probability of a correct response to an item as a function of the latent attributes of a respondent

\[ \alpha = 0 \quad \alpha = 1 \quad P(X=1) \]

• The latent attributes are categorical, meaning a respondent can have one of countably many possible statuses
  ➢ Each status corresponds to a predicted probability of a correct response

• As such, the LCDM is very similar to an ANOVA model
  ➢ Predicting the a dependent variable as a function of the experimental group of a respondent

\[ \alpha = 0 \quad \alpha = 1 \quad P(X=1) \]
• As a refresher on ANOVA, let's imagine that we are interested in the factors that have an effect on work output (denoted by Y)

• We design a two-factor study where work output may be affected by:
  ➢ Lighting of the workplace
    ✷ High or Low
  ➢ Temperature
    ✷ Cold or Warm

• This experimental design is known as a 2-Way ANOVA
Here is the 2 x 2 Factorial design:

<table>
<thead>
<tr>
<th>Low Lighting</th>
<th>High Lighting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cold Temperature</td>
<td>$\bar{Y}_{\text{Cold,Low}}$</td>
</tr>
<tr>
<td>Warm Temperature</td>
<td>$\bar{Y}_{\text{Warm,Low}}$</td>
</tr>
</tbody>
</table>

The ANOVA model for a respondent’s work output is

$$Y_r = \mu + A_t + B_l + (AB)_{tl} + \varepsilon_r$$
• The ANOVA model allows us to test for the presence of
  - A main effect associated with Temperature \((A_t)\)
    - Where \(A_{\text{Cold}} + A_{\text{Warm}} = 0\)
  - A main effect associated with Lighting \((B_l)\)
    - Where \(B_{\text{Low}} + B_{\text{High}} = 0\)
  - An interaction effect associated with Temperature and Lighting \((AB)_{tl}\)
    - Where \((AB)_{\text{Cold,Low}} + (AB)_{\text{Cold,High}} + (AB)_{\text{Warm,Low}} + (AB)_{\text{Warm,High}} = 0\)

\[
Y_r = \mu + A_t + B_l + (AB)_{tl} + \varepsilon_r
\]
The ANOVA model can also be re-written using two dummy-coded variables.

\[ Warm_r = \]
- 0 for respondents in \textit{cold temperature} condition
- 1 for respondents in \textit{warm temperature} condition

\[ High_r = \]
- 0 for respondents in \textit{low lighting} condition
- 1 for respondents in \textit{high lighting} condition
ANOVA with Dummy Coded Variables

- The ANOVA model then becomes:

\[
Y_r = \beta_0 + \beta_t \text{Warm}_r + \beta_l \text{High}_r + \beta_{t*l} \text{Warm}_r \times \text{High}_r + \epsilon_r
\]

<table>
<thead>
<tr>
<th>Warm(_r) = 0 Cold Temperature</th>
<th>Warm(_r) = 1 Warm Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low Lighting  (\bar{Y}_{\text{Cold,Low}})</td>
<td>High Lighting (\bar{Y}_{\text{Cold,High}})</td>
</tr>
<tr>
<td>(\bar{Y}_{\text{Warm,Low}})</td>
<td>(\bar{Y}_{\text{Warm,High}})</td>
</tr>
</tbody>
</table>
ANOVA Effects Explained

\[ Y_r = \beta_0 + \beta_t Warm_r + \beta_l High_r + \beta_{t*l} Warm_r * High_r + \varepsilon_r \]

- \(\beta_0\) is the mean for the cold and low light condition (reference group)
  - The intercept

- \(\beta_t\) is the difference in the average response for warm temperature for a business with low lights \(High_r = 0\) (Conditional Main Effect)

- \(\beta_l\) is the difference in the average response for high lights for a business with cold temperature \(Warm_r = 0\) (Conditional Main Effect)

- \(\beta_{t*l}\) is additional change in average that is not explained by the shift in temperature and shift and lights, when both occur (2-Way Interaction)

- Respondents from in the same condition have the same predicted value
• The ANOVA model and the LCDM take the same modeling approach
  ➢ Predict a response using dummy coded variables
    ♦ In LCDM dummy coded variables are latent attributes
  ➢ Using a set of main effects and interactions
    ♦ Links attributes to item response
  ➢ Where possible, we may look for ways to reduce the model
    ♦ Removing non-significant interactions and/or main effects
Differences Between LCDM and ANOVA

• The LCDM and the ANOVA model differ in two ways:
  - Instead of a continuous outcome such as work output the LCDM models a function of the probability of a correct response
    ▪ The logit of a correct response (defined next)
  - Instead of observed “factors” as predictors the LCDM uses discrete *latent* variables (the attributes being measured)

• Attributes are given dummy codes (act as latent factors)
  - $\alpha_{ra} = 1$ if respondent $r$ has *mastered* attribute $a$
  - $\alpha_{ra} = 0$ if respondent $r$ has *not mastered* attribute $a$

• The LCDM treats the attributes as *crossed* experimental factors: all combinations are assumed to exist
  - This assumption can be (and will be) modified
LOGITS EXPLAINED
The LCDM models the log-odds of a correct response conditional on a respondent’s attribute pattern $\alpha_r$.

- The log-odds is called a logit

$$
Logit(Y_{ri} = 1|\alpha_r) = \log\left(\frac{P(Y_{ri} = 1|\alpha_r)}{1 - P(Y_{ri} = 1|\alpha_r)}\right)
$$

- Here $\log(\cdot)$ is the natural log

The logit is used because the responses are binary.
- Items are either answered correctly (1) or incorrectly (0)

The linear model with an identity link and Gaussian error is inappropriate for categorical data.
- Can lead to impossible predictions (i.e., probabilities greater than 1 or less than 0)
More on Logits

![Logit-Probability Graph]

<table>
<thead>
<tr>
<th>Probability</th>
<th>Logit</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.0</td>
</tr>
<tr>
<td>0.9</td>
<td>2.2</td>
</tr>
<tr>
<td>0.1</td>
<td>-2.2</td>
</tr>
<tr>
<td>0.99</td>
<td>4.6</td>
</tr>
</tbody>
</table>
From Logits to Probabilities

• Whereas logits are useful as they are unbounded continuous variables, categorical data analyses rely on estimated probabilities

• The inverse logit function converts the unbounded logit to a probability

  ➢ This is also the form of an IRT model (and logistic regression)

  \[ P(Y_{ri} = 1|\alpha_r) = \frac{\exp(\text{Logit}(Y_{ri} = 1|\alpha_r))}{1 + \exp(\text{Logit}(Y_{ri} = 1|\alpha_r))} \]

  ➢ Here, \( \exp(\cdot) = 2.718282 \): the inverse function of the natural log (Euler’s number)

  ➢ Sometimes this is written:

  \[ P(Y_{ri} = 1|\alpha_r) = \frac{\exp(\text{Logit}(Y_{ri} = 1|\alpha_r))}{1 + \exp(\text{Logit}(Y_{ri} = 1|\alpha_r))} = \left[1 + \exp(-\text{Logit}(Y_{ri} = 1|\alpha_r))\right]^{-1} \]
THE LCDM
To demonstrate the LCDM, consider the item 2+3-1=? from our basic math example

- Measures addition (attribute 1: $\alpha_{r1}$) and subtraction (attribute 2: $\alpha_{r2}$)

Only attributes defined by the Q-matrix are modeled for an item

The LCDM provides the logit of a correct response as a function of the latent attributes mastered by a respondent:

$$\text{Logit}(Y_{ri} = 1|\alpha_r) = \lambda_{i,0} + \lambda_{i,1,(1)}\alpha_{r1} + \lambda_{i,1,(2)}\alpha_{r2} + \lambda_{i,2,(1,2)}\alpha_{r1}\alpha_{r2}$$
Logit($Y_{ri} = 1|\alpha_r$) = $\lambda_{i,0} + \lambda_{i,1,(1)}\alpha_{r1} + \lambda_{i,1,(2)}\alpha_{r2} + \lambda_{i,2,(1,2)}\alpha_{r1}\alpha_{r2}$

- $Logit(Y_{ri} = 1|\alpha_r)$ is the logit of a correct response to item $i$ by respondent $r$

- $\lambda_{i,0}$ is the intercept
  - The logit for non-masters of addition and subtraction
  - The reference group is respondents who have not mastered either attribute ($\alpha_{r1} = 0$ and $\alpha_{r2} = 0$)
\[ \text{Logit}(Y_{ri} = 1|\alpha_r) = \lambda_{i,0} + \lambda_{i,1,(1)} \alpha_{r1} + \lambda_{i,1,(2)} \alpha_{r2} + \lambda_{i,2,(1,2)} \alpha_{r1} \alpha_{r2} \]

- \( \lambda_{i,1,(1)} \) = **conditional main effect** for addition (attribute 1)  
  ➢ The increase in the logit for mastering addition (for someone who has **not** mastered subtraction)

- \( \lambda_{i,1,(2)} \) = **conditional main effect** for subtraction (attribute 2)  
  ➢ The increase in the logit for mastering subtraction (for someone who has **not** mastered addition)

- \( \lambda_{i,2,(1,2)} \) = **is the 2-way interaction** between addition and subtraction (attributes 1 and 2)  
  ➢ Change in the logit for mastering **both** addition & subtraction
Understanding LCDM Notation

- The LCDM item parameters have several subscripts:
  \[ \lambda_{i,e,(a_1,...)} \]

- Subscript #1 – \( i \): the item to which parameters belong

- Subscript #2 – \( e \): the level of the effect
  - 0 is the intercept
  - 1 is the main effect
  - 2 is the two-way interaction
  - 3 is the three-way interaction

- Subscript #3 – \((a_1,...)\): the attributes to which the effect applies
  - Same number of attributes listed as number in Subscript #2
LCDM: A NUMERICAL EXAMPLE
• Imagine we obtained the following estimates for the item $2 + 3 - 1 = ?$:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Effect Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{i,0}$</td>
<td>-2</td>
<td>Intercept</td>
</tr>
<tr>
<td>$\lambda_{i,1,(1)}$</td>
<td>2</td>
<td>Addition Conditional Main Effect</td>
</tr>
<tr>
<td>$\lambda_{i,1,(2)}$</td>
<td>1</td>
<td>Subtraction Conditional Main Effect</td>
</tr>
<tr>
<td>$\lambda_{i,2,(1,2)}$</td>
<td>0</td>
<td>Addition/Subtraction Interaction</td>
</tr>
</tbody>
</table>
### LCDM Predicted Logits and Probabilities

<table>
<thead>
<tr>
<th>$\alpha_{r1}$</th>
<th>$\alpha_{r2}$</th>
<th>LCDM Logit Function</th>
<th>Logit</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$\lambda_{i,0} + \lambda_{i,1,(1)} \times (0) + \lambda_{i,1,(2)} \times (0) + \lambda_{i,2,(1,2)} \times (0) \times (0)$</td>
<td>-2</td>
<td>0.12</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>$\lambda_{i,0} + \lambda_{i,1,(1)} \times (0) + \lambda_{i,1,(2)} \times (1) + \lambda_{i,2,(1,2)} \times (0) \times (1)$</td>
<td>-1</td>
<td>0.27</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$\lambda_{i,0} + \lambda_{i,1,(1)} \times (1) + \lambda_{i,1,(2)} \times (0) + \lambda_{i,2,(1,2)} \times (1) \times (0)$</td>
<td>0</td>
<td>0.50</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\lambda_{i,0} + \lambda_{i,1,(1)} \times (1) + \lambda_{i,1,(2)} \times (1) + \lambda_{i,2,(1,2)} \times (1) \times (1)$</td>
<td>1</td>
<td>0.73</td>
</tr>
</tbody>
</table>

**Logit Response Function**

**Probability Response Function**

(Item Characteristic Bar Chart)
The LCDM interaction term can be investigated via plots.

- **No interaction**: parallel lines for the logit
  - Compensatory RUM (Hartz, 2002)
Strong Positive Interactions

- **Positive interaction**: over-additive logit model
  - Conjunctive model (i.e., all-or-none)
  - DINA model (Haertel, 1989; Junker & Sijtsma, 1999)

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Logit Response Function

Probability Response Function

(Item Characteristic Bar Chart)
**Strong Negative Interactions**

- **Negative interaction:** under-additive logit model
  - Disjunctive model (i.e., one-or-more)
  - DINO model (Templin & Henson, 2006)

![Logit Response Function](image)

![Probability Response Function](image)

**Possible Attribute Patterns**

- $\alpha_1=0; \alpha_2=0$
- $\alpha_1=0; \alpha_2=1$
- $\alpha_1=1; \alpha_2=0$
- $\alpha_1=1; \alpha_2=1$
Less Extreme Interactions

- Extreme interactions are unlikely in practice
- Below: positive interaction with positive main effects

Logit Response Function

Probability Response Function (Item Characteristic Bar Chart)
GENERAL FORM OF THE LCDM
More General Versions of the LCDM

• The LCDM is based on the General Diagnostic Model by von Davier (GDM; 2005)
  ➢ The GDM allows for both categorical and continuous latent variables

• For items measuring more than two attributes, higher level interactions are possible
  ➢ Difficult to estimate in practice

• The LCDM appears in the psychometric literature in a more general form
  ➢ See Henson, Templin, & Willse (2009)
General Form of the LCDM

- The LCDM specifies the probability of a correct response as a function of a set of attributes and a Q-matrix:

\[
P(Y_{ri} = 1|\alpha_r) = \frac{\exp\left(\lambda_i^T h(q_i, \alpha_r)\right)}{1 + \exp\left(\lambda_i^T h(q_i, \alpha_r)\right)}
\]
Unpacking the General Form of the LCDM: Components $\alpha_r$ and $q_i$

- The key to understanding the general form of the LCDM is to understand that it is a general equation that makes any possible number of attributes be measured by an item

- To put this into context, we will continue with our basic mathematics example
  
  - Overall four attributes measured: Addition ($\alpha_{r1}$), Subtraction ($\alpha_{r2}$), Multiplication ($\alpha_{r3}$), Division ($\alpha_{r4}$)
    - Attribute profile vector for respondent $r$: $\alpha_r = [\alpha_{r1} \  \alpha_{r2} \  \alpha_{r3} \  \alpha_{r4}]$ (size 1 x 4)
  
  - Item $i$: $2 + 3 - 1 = ?$
    - Measures Addition ($\alpha_{r1}$) and Subtraction ($\alpha_{r2}$)
    - Q-matrix row vector for item $i$: $q_i = [1 \  1 \  0 \  0]$
Unpacking the General Form of the LCDM: Parameter Vector $\lambda_i$

- From the general LCDM notation, $\lambda_i$ is a vector of all possible item parameters for item $i$
  - All possible: if all $A$ Q-matrix entries in $q_i$ were equal to 1 (so size is $2^A \times 1$)
  - Not all parameters will be estimated if some $q_{ia} = 0$

- For four-attribute example,

$$\lambda_i = \begin{bmatrix}
\lambda_{i,0} \\
\lambda_{i,1,(1)} \\
\lambda_{i,1,(2)} \\
\lambda_{i,1,(3)} \\
\lambda_{i,1,(4)} \\
\lambda_{i,2,(1,2)} \\
\lambda_{i,2,(1,3)} \\
\lambda_{i,2,(1,4)} \\
\lambda_{i,2,(2,3)} \\
\lambda_{i,2,(2,4)} \\
\lambda_{i,2,(3,4)} \\
\lambda_{i,3,(1,2,3)} \\
\lambda_{i,3,(1,2,4)} \\
\lambda_{i,3,(1,3,4)} \\
\lambda_{i,3,(2,3,4)} \\
\lambda_{i,4,(1,2,3,4)} \\
\end{bmatrix}_{(16 \times 1)}$$

- Intercept
- Possible Main Effects
- Possible Two-Way Interactions
- Possible Higher-Order Interactions
Unpacking the General Form of the LCDM: Helper Function \( h(q_i, \alpha_r) \)

- From the general LCDM notation \( h(q_i, \alpha_r) \) is a vector-valued function
  - Vector valued = result is a vector

<table>
<thead>
<tr>
<th>( h(q_i, \alpha_r) )</th>
<th>( \lambda_i )</th>
</tr>
</thead>
</table>
| \[ 1 
    (q_{i1}\alpha_{r1}) 
    (q_{i2}\alpha_{r2}) 
    (q_{i3}\alpha_{r3}) 
    (q_{i4}\alpha_{r4}) 
    (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2}) 
    (q_{i1}\alpha_{r1})(q_{i3}\alpha_{r3}) 
    (q_{i1}\alpha_{r1})(q_{i4}\alpha_{r4}) 
    (q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3}) 
    (q_{i2}\alpha_{r2})(q_{i4}\alpha_{r4}) 
    (q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) 
    (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3}) 
    (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2})(q_{i4}\alpha_{r4}) 
    (q_{i1}\alpha_{r1})(q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) 
    (q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) 
| \[ \lambda_i,0 
    \lambda_i,1,(1) 
    \lambda_i,1,(2) 
    \lambda_i,1,(3) 
    \lambda_i,1,(4) 
    \lambda_i,2,(1,2) 
    \lambda_i,2,(1,3) 
    \lambda_i,2,(1,4) 
    \lambda_i,2,(2,3) 
    \lambda_i,2,(2,4) 
    \lambda_i,2,(3,4) 
    \lambda_i,3,(1,2,3) 
    \lambda_i,3,(1,2,4) 
    \lambda_i,3,(1,3,4) 
    \lambda_i,3,(2,3,4) 
| } 

\( h(q_i, \alpha_r) = \text{ (16x1) } \)

\( \lambda_i = \text{ (16x1) } \)
More on the Helper Function $h(q_i, \alpha_r)$

- For a specific item $i$ with a specific Q-matrix row vector $q_i$, the role of the helper function $h(q_i, \alpha_r)$ becomes more transparent

\[
 h(q_i, \alpha_r) = \begin{bmatrix}
 1 \\
 (q_{i1}\alpha_1) = \alpha_{r1} \\
 (q_{i2}\alpha_2) = \alpha_{r2} \\
 (q_{i3}\alpha_3) = 0 \\
 (q_{i4}\alpha_4) = 0 \\
 (q_{i1}\alpha_1)(q_{i2}\alpha_2) = \alpha_{r1}\alpha_{r2} \\
 (q_{i1}\alpha_1)(q_{i3}\alpha_3) = 0 \\
 (q_{i1}\alpha_1)(q_{i4}\alpha_4) = 0 \\
 (q_{i2}\alpha_2)(q_{i3}\alpha_3) = 0 \\
 (q_{i2}\alpha_2)(q_{i4}\alpha_4) = 0 \\
 (q_{i3}\alpha_3)(q_{i4}\alpha_4) = 0 \\
 (q_{i1}\alpha_1)(q_{i2}\alpha_2)(q_{i3}\alpha_3) = 0 \\
 (q_{i1}\alpha_1)(q_{i2}\alpha_2)(q_{i4}\alpha_4) = 0 \\
 (q_{i1}\alpha_1)(q_{i3}\alpha_3)(q_{i4}\alpha_4) = 0 \\
 (q_{i2}\alpha_2)(q_{i3}\alpha_3)(q_{i4}\alpha_4) = 0 \\
 (q_{i1}\alpha_1)(q_{i2}\alpha_2)(q_{i3}\alpha_3)(q_{i4}\alpha_4) = 0 \\
\end{bmatrix}_{(16\times1)}
\]

\[
 \lambda_i = \begin{bmatrix}
 \lambda_{i,0} \\
 \lambda_{i,1,(1)} \\
 \lambda_{i,1,(2)} \\
 \lambda_{i,1,(3)} \\
 \lambda_{i,1,(4)} \\
 \lambda_{i,2,(1,2)} \\
 \lambda_{i,2,(1,3)} \\
 \lambda_{i,2,(1,4)} \\
 \lambda_{i,2,(2,3)} \\
 \lambda_{i,2,(2,4)} \\
 \lambda_{i,2,(3,4)} \\
 \lambda_{i,3,(1,2,3)} \\
 \lambda_{i,3,(1,2,4)} \\
 \lambda_{i,3,(1,3,4)} \\
 \lambda_{i,3,(2,3,4)} \\
 \lambda_{i,4,(1,2,3,4)} \\
\end{bmatrix}_{(16\times1)}
\]
Putting It All Together: The Matrix Product $\lambda_i^T h(q_i, \alpha_r)$

- The term in the exponent is the logit we have been using all along:

  \[
  \logit(Y_{ri} = 1 | \alpha_r) = \lambda_i,0 + \sum_{a=1}^{A} \lambda_{i,1,(a)}(q_{ia}\alpha_{ra}) + \sum_{a=1}^{A-1} \sum_{b=a+1}^{A} \lambda_{i,2,(a,b)}(q_{ia}\alpha_{ra})(q_{ib}\alpha_{rb}) + \ldots
  \]

  - **Intercept**
  - **Main Effects**
  - **Two-Way Interactions**
  - **Higher Interactions**

- For our example item:

  \[
  (\lambda_i^T)_{1 \times 16} h(q_i, \alpha_r)_{16 \times 1} = \lambda_i,0 + \lambda_{i,1,(1)}\alpha_{r1} + \lambda_{i,1,(2)}\alpha_{r2} + \lambda_{i,2,(1,2)}\alpha_{r1}\alpha_{r2}
  \]

  - Result is a scalar $(1 \times 1)$
SUBSUMED MODELS
Previously Popular DCMs

- Because the advent of the GDM and LCDM has been fairly recent, other earlier DCMs are still in use

- Such DCMs are much more restrictive than the LCDM
  - Not discussed at length here
  - It is anticipated that field will adapt to more general forms

- Each of these models can be fit using the LCDM
  - Fixing certain model parameters

- Shown for reference purposes
  - See Henson, Templin, & Willse (2009) for more detail
Other DCMs with the LCDM

• The Big 6 - DCMs with latent variables:
  - **DINA** (Deterministic Inputs, Noisy ‘AND’ Gate)
    - Haertel (1989); Junker and Sijtsma (1999)
  - **NIDA** (Noisy Inputs, Deterministic ‘AND’ Gate)
  - **RUM** (Reparameterized Unified Model)
    - Hartz (2002)
  - **DINO** (Deterministic Inputs, Noisy ‘OR’ Gate)
    - Templin & Henson (2006)
  - **NIDO** (Noisy Inputs, Deterministic ‘OR’ Gate)
    - Templin (2006)
  - **C-RUM** (Compensatory Reparameterized Unified Model)
    - Hartz (2002)
### Other DCMs with the LCDM

<table>
<thead>
<tr>
<th>LCDM Parameters</th>
<th>Non-compensatory Models</th>
<th>Compensatory Models</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DINA</td>
<td>NIDA</td>
</tr>
<tr>
<td>Main Effects</td>
<td>Zero</td>
<td>Positive</td>
</tr>
<tr>
<td>Interactions</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>Parameter Restrictions</td>
<td>Across Attributes</td>
<td>Across Items</td>
</tr>
</tbody>
</table>

Adapted from: Rupp, Templin, and Henson (2010)
Compensatory RUM (Hartz, 2002)

- No interactions in model
- No interaction: parallel lines for the logit

\[
\text{Logit}(Y_{ri} = 1|\alpha_{r}) = \lambda_{i,0} + \lambda_{i,1,(1)}\alpha_{r1} + \lambda_{i,1,(2)}\alpha_{r2}
\]
• **Positive interaction**: over-additive logit model
  - Highest interaction parameter is non-zero
  - All main effects (and lower interactions) zero

\[
\text{Logit}(Y_{ri} = 1|\alpha_r) = \lambda_{i,0} + \lambda_{i,2(1,2)}\alpha_{r1}\alpha_{r2}
\]
DINO Model (Templin & Henson, 2006)

- **Negative interaction**: under-additive logit model
  - All main effects equal
  - Interaction terms are $-1$ sum of corresponding lower effects

$$\text{Logit}(Y_{ri} = 1|\alpha_r) = \lambda_{i,0} + \lambda_{i,1} \alpha_{r1} + \lambda_{i,1} \alpha_{r2} - \lambda_{i,1} \alpha_{r1} \alpha_{r2}$$

---

**Logit Response Function**

**Item Characteristic Bar Chart**

[Graphs showing logit response function and item characteristic bar chart]
EXAMPLE RESULTS: DTMR PROJECT
• Referent unit ($\alpha_1$) and partitioning and iterating ($\alpha_2$) are measured
• Q-matrix entries:

<table>
<thead>
<tr>
<th></th>
<th>RU</th>
<th>PI</th>
<th>APP</th>
<th>MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Item 22</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

• LCDM item response function:

$$logit(X_{ei} = 1|\alpha_e) = \lambda_{i,0} + \lambda_{i,1,(1)}\alpha_{e1} + \lambda_{i,1,(2)}\alpha_{e2} + \lambda_{i,2,(1,2)}\alpha_{e1}\alpha_{e2}$$

Intercept  Main Effect (RU)  Main Effect (PI)  Interaction (Between RU and PI)
### Table 1. DTMR Item Parameter Estimates

<table>
<thead>
<tr>
<th>i</th>
<th>$\lambda_{i,0}$</th>
<th>RU($\alpha_1$)</th>
<th>PI($\alpha_2$)</th>
<th>APP($\alpha_3$)</th>
<th>MC($\alpha_4$)</th>
<th>RU/PI</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-1.12$ (0.12)</td>
<td>2.24 (0.20)</td>
<td>1.70 (0.24)</td>
<td>1.52 (0.20)</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.59 (0.13)</td>
<td></td>
<td>1.27 (0.22)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$-2.07$ (0.22)</td>
<td>0.65 (0.19)</td>
<td></td>
<td>2.08 (0.50)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$-1.19$ (0.11)</td>
<td>1.52 (0.20)</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$-1.67$ (0.14)</td>
<td>1.20 (0.22)</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$-3.81$ (0.47)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$-0.73$ (0.09)</td>
<td></td>
<td></td>
<td>1.81 (0.21)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8a</td>
<td>$-0.62$ (0.25)</td>
<td>4.25 (0.64)</td>
<td>2.16 (0.24)</td>
<td>4.84 (0.55)</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>8b</td>
<td>$-0.09$ (0.17)</td>
<td></td>
<td>0.87 (0.18)</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>8c</td>
<td>0.28 (0.13)</td>
<td></td>
<td></td>
<td>1.89 (0.21)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8d</td>
<td>$-1.03$ (0.17)</td>
<td>0.76 (0.19)</td>
<td>4.26 (0.73)</td>
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</tr>
<tr>
<td>9</td>
<td>$-1.22$ (0.10)</td>
<td></td>
<td>4.57 (0.87)</td>
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<tr>
<td>10a</td>
<td>$-0.50$ (0.18)</td>
<td>1.32 (0.28)</td>
<td>4.84 (0.55)</td>
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</tr>
<tr>
<td>10b</td>
<td>$-4.01$ (0.74)</td>
<td>1.30 (0.26)</td>
<td>2.94 (0.28)</td>
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</tr>
<tr>
<td>10c</td>
<td>$-4.89$ (0.87)</td>
<td></td>
<td>3.04 (0.31)</td>
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<tr>
<td>11</td>
<td>$-0.88$ (0.01)</td>
<td>1.25 (0.18)</td>
<td>1.05 (0.28)</td>
<td>*</td>
<td>*</td>
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<tr>
<td>12</td>
<td>$-1.29$ (0.11)</td>
<td>1.89 (0.21)</td>
<td>1.05 (0.28)</td>
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<td>*</td>
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<tr>
<td>13</td>
<td>$-0.74$ (0.14)</td>
<td>0.45 (0.20)</td>
<td>1.22 (0.27)</td>
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<tr>
<td>14</td>
<td>$-2.14$ (0.14)</td>
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<td>1.10 (0.24)</td>
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<tr>
<td>15a</td>
<td>$-2.48$ (0.29)</td>
<td>2.72 (0.26)</td>
<td>1.59 (0.21)</td>
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<tr>
<td>15b</td>
<td>$-0.56$ (0.18)</td>
<td>2.94 (0.28)</td>
<td>1.27 (0.34)</td>
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<tr>
<td>15c</td>
<td>$-0.44$ (0.17)</td>
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<td>1.27 (0.34)</td>
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<tr>
<td>16</td>
<td>$-0.86$ (0.01)</td>
<td>1.55 (0.23)</td>
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</tr>
<tr>
<td>17</td>
<td>$-2.08$ (0.23)</td>
<td>1.22 (0.27)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>$-0.99$ (0.14)</td>
<td>1.13 (0.26)</td>
<td>1.05 (0.28)</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>19</td>
<td>$-1.50$ (0.13)</td>
<td>1.69 (0.19)</td>
<td>1.41 (0.24)</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>20</td>
<td>$-2.28$ (0.16)</td>
<td></td>
<td>1.43 (0.25)</td>
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</tr>
<tr>
<td>21</td>
<td>$-2.08$ (0.23)</td>
<td>1.22 (0.27)</td>
<td>1.27 (0.34)</td>
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<td></td>
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<tr>
<td>22</td>
<td>$-0.99$ (0.14)</td>
<td>1.13 (0.26)</td>
<td>1.41 (0.24)</td>
<td></td>
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</tr>
</tbody>
</table>

*Note.* Standard errors for parameters are given in parenthesis. Item 20 was removed due to scoring. Asterisks (*) indicates the parameter was estimated in the initially hypothesized parameterization.
CONCLUDING REMARKS
Wrapping Up – Lecture Take-Home Points

• The LCDM uses an ANOVA-like approach to map latent attributes onto item responses
  - Uses main effects and interactions for each attribute
  - Uses a logit link function

• Multiple diagnostic models are subsumed by the LCDM