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# **Random Slopes, Cross-level Interactions, and Analysis Interpretations**

Applied Multilevel Models for Cross-Sectional Data  
Lecture 7

ICPSR Summer Workshop  
University of Colorado Boulder

# Covered this Section

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- Reinforcing MLM through examples:
  - Random slopes
  - Cross-level interactions
- A new example with actual data
  - Class-guided analysis

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**AS SEEN LAST TIME...**

# Guiding Example

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- Imagine you are interested in studying the effects of socioeconomic status (SES) on student achievement
  - What do you think the relationship between student achievement and SES happens to be?
- You are interested in predicting achievement from SES
  - Your guiding research question

# Your Study...

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- Let's imagine you are able to get data from 7 elementary schools around Boulder
  - You sample 50 students from each elementary school
  - You record a measure of their SES (scale with a mean of 50)
  - You record a measure of their achievement (scale with a mean of 100)
  
- Both scales magically have absolutely perfect reliability

## Previously Called Model #3...

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- Adding cluster-mean centered student SES and grand-mean centered school mean SES yields the following model:
- Level-1:  $Y_{is} = \beta_{0s} + \beta_{1s}(X_{is} - \bar{X}_s) + e_{is}$ 
  - Where  $e_{is} \sim N(0, \sigma^2)$
- Level-2:  $\beta_{0s} = \gamma_{00} + \gamma_{01}(\bar{X}_s - \bar{X}_g) + U_{0s}$   
 $\beta_{1s} = \gamma_{10}$ 
  - Where  $U_{0s} \sim N(0, \tau_0^2)$
- Combined:

$$Y_{is} = \gamma_{00} + \gamma_{01}(\bar{X}_s - \bar{X}_g) + \gamma_{10}(X_{is} - \bar{X}_s) + U_{0s} + e_{is}$$

# Model #3 Results- Model Fit

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- Model Fit:

Model #2 (old model)

Value	Estimate
Deviance	2,202.5

Model #3 (new)

Value	Estimate
Deviance	2,147.9

- Question: Is model 3 preferred to model 2?
- Answer: Deviance test (two new parameters:
  - Test statistic:  $2,202.5 - 2,147.9 = 57.6$
  - Degrees of freedom = 2 (new parameters  $\gamma_{10}$  and  $\gamma_{01}$ )
  - P-value:  $< 0.0001$
- Conclusion: Model #3 is preferred

# Model #3 Results – Fixed Effects (Means)

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- Model parameter estimates:
  - $\gamma_{00} = 101.38 (1.41); p < 0.0001$ 
    - ◆ The overall intercept – the value of achievement for a student with SES equal to their school mean SES at a school with mean SES equal to the grand mean SES
    - ◆ Is the average value of achievement
  - $\gamma_{10} = -0.99 (0.14); p < 0.0001$ 
    - ◆ The slope for student SES (minus school mean SES)
    - ◆ Represents the change in achievement for each unit a student SES differs from their school mean
      - Given school mean SES is held constant
  - $\gamma_{01} = 1.63 (0.42); p = 0.0060$ 
    - ◆ The slope for school mean SES (minus grand mean SES)
    - ◆ Represents the change in achievement for each unit the school mean SES differs from the grand mean
      - Given student SES is held constant



# Model #3 Results – Variance Parameters

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- Our estimated variance parameters were:

Parameter	Estimate
$\sigma^2$ (Error)	25.35 (1.93)
$\tau_0^2$ (Random Intercept)	13.35 (7.41)

- Meaning, our intraclass correlation was:

$$\rho = \frac{13.35}{13.35 + 25.35} = .345$$

- This means:
  - Student's achievement scores within a school had a correlation of .345
  - 34.5% of the total variability in achievement scores came from between school variability
  - Our linear regression assumption of uncorrelated residuals is violated
    - ◆ Should be using the mixed model with a random intercept

# More on Variances

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- In comparing Model #3 to Model #2, an important distinction is of how much each variance component is reduced because of addition of the predictors
  - Called a pseudo-  $R^2$
- Level-2 Variance ( $\tau_0^2$ ):
  - Model #2 = 43.25; Model #3 = 13.35
  - Reduction:  $43.25 - 13.35 = 29.9$
  - Proportion of Model #2 Variance Explained:  $29.9/43.25 = .69$
  - Explanation: School Mean SES explains 69% of the variance in school achievement (random intercept variance)

# Level-1 Variance Reduction

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- Level-1 Variance ( $\sigma^2$ ):
  - Model #2 = 29.04; Model #3 = 25.35
  - Reduction:  $29.04 - 25.35 = 3.69$
  - Proportion of Model #2 Variance Explained:  $3.69/29.04 = .13$
  - Explanation: Student SES explains 13% of the variance in student achievement

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**UP NEXT: RANDOM SLOPES**

# More On Our Analysis

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- So far, we've been concerned that there are differences in the relationship between SES and achievement at the differing levels of our analysis
  - Between-schools: Positive relationship
  - Within-schools: Negative relationship
- We also have investigated whether or not schools have differences in their intercepts
  - Random intercept variance was non-zero
  - Was able to be predicted by school-level SES
- But...what about the slope for SES
  - Shouldn't we see if there are school-level differences in slope?

## Model #4...Adding a Random Slope

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- We will now expand our Level-2 Model to include a random effect
  - Called a random slope
- We will add a level-2 predictor of the slope, too...
  - Leading to a cross-level interaction.

- Level-1:  $Y_{is} = \beta_{0s} + \beta_{1s}(X_{is} - \bar{X}_s) + e_{is}$ 
  - Where  $e_{is} \sim N(0, \sigma^2)$

- Level-2:  $\beta_{0s} = \gamma_{00} + \gamma_{01}(\bar{X}_s - \bar{X}_g) + U_{0s}$   
 $\beta_{1s} = \gamma_{10} + U_{1s}$

- Where  $\begin{bmatrix} U_{0s} \\ U_{1s} \end{bmatrix} \sim N_2 \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{bmatrix} \right)$

- Combined:

$$Y_{is} = \gamma_{00} + \gamma_{01}(\bar{X}_s - \bar{X}_g) + \gamma_{10}(X_{is} - \bar{X}_s) + U_{0s} + U_{1s}(X_{is} - \bar{X}_s) + e_{is}$$

# The Process

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- Just as with our transition from or previous models, we must first determine if the addition of a random slope (and accompanying covariance between random intercept and slope) are needed in the model.

Model #3 (Old Model)

Value	Estimate
Deviance	2,147.9

Model #4 (Random Slope)

Value	Estimate
Deviance	2,143.8

- Test statistic:  $2,147.9 - 2,143.8 = 3.9$
- Degrees of freedom = 2 (new parameters  $\tau_{01}$  and  $\tau_1^2$ )
- P-value:  $p = 0.1423$
- Conclusion: No variation in slopes across schools

## So...No Random Slopes

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- Our analysis resulted in not finding any variation in the random slopes
  - Model #3 (just random intercept) was preferred to Model #4 (random intercepts + random slopes)
- The Model #4 estimates for the random effect variance components were:
  - $\tau_0^2 = 20.47$
  - $\tau_1^2 = 0$
  - $\tau_{01} = -1.34^*$ 
    - ◆ Note this covariance is not possible (think of correlation)
- So...schools do not vary in how student SES is related to student achievement



# One More Analysis...Investigating Cross-Level Interactions

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- One question remains: does the effect of student SES on achievement vary as a function of school-level SES
  - Perhaps effect is stronger in lower-SES schools?
- This would be what is considered a **cross-level interaction**
  - Level-2 (school SES) and level-1 (student SES) variables interacting to produce an effect on outcome (achievement)
- Usually is tested when random slope is present...
  - ...but can be without (different df used)

# Model #5...Adding a Cross-Level Interaction

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- We will now expand our Level-2 Model to include a random effect
  - Called a random slope
- We will add a level-2 predictor of the slope, too...
  - Leading to a cross-level interaction.

- Level-1:  $Y_{is} = \beta_{0s} + \beta_{1s}(X_{is} - \bar{X}_s) + r_{is}$ 
  - Where  $r_{is} \sim N(0, \sigma^2)$

- Level-2:  $\beta_{0s} = \gamma_{00} + \gamma_{01}(\bar{X}_s - \bar{X}_g) + U_{0s}$   
 $\beta_{1s} = \gamma_{10} + \gamma_{11}(\bar{X}_s - \bar{X}_g)$

- Where  $U_{0s} \sim N(0, \tau_0^2)$

- Combined:

$$Y_{is} = \gamma_{00} + \gamma_{01}(\bar{X}_s - \bar{X}_g) + \gamma_{10}(X_{is} - \bar{X}_s) +$$

# The Process

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## Model #3 (Old Model)

Value	Estimate
Deviance	2,147.9

## Model #5 (Cross-Level Interaction)

Value	Estimate
Deviance	2,147.4

- Test statistic:  $2,147.9 - 2,147.40 = 0.5$
- Degrees of freedom = 1 (new parameter:  $\gamma_{11}$ )
- P-value:  $p = 0.4795$
- Conclusion: No cross-level interaction present