Latent Trait Measurement Models for Binary Responses: IRT and IFA

- Today's topics:
  - The Big Picture of Measurement Models
  - 1, 2, 3, and 4 Parameter IRT (and Rasch) Models
  - Item and Test Information
  - Item Response Models → Item Factor Models
  - Model Estimation, Comparison, and Evaluation

The Big Picture of CTT

- **CTT** predicts the sum score: $\mathbf{Y_s = TrueScore_s + e_s}$
  - Items are assumed exchangeable, and their properties are not part of the model for creating a latent trait estimate
  - **Because the latent trait estimate IS the sum score**, it is problematic to make comparisons across different test forms
    - Item difficulty = mean of item (is sample-dependent)
    - Item discrimination = item-total correlation (is sample-dependent)
  - Estimates of reliability assume (without testing) unidimensionality and tau-equivalence (alpha) or parallel items (Spearman-Brown)
    - Measurement error is assumed *constant* across the trait level (one value)

- How do you make your test better?
  - Get more items. What kind of items? More.

The Big Picture of CFA

- **CFA predicts the ITEM response:** $\mathbf{y_{is} = \mu_i + \lambda_i F_s + e_{is}}$
  - **Linear regression relating continuous item response to latent predictor F**
  - Both items AND subjects matter in predicting responses
    - Item difficulty = intercept $\boldsymbol{\mu_i}$ (in theory, sample independent)
    - Item discrimination = factor loading $\boldsymbol{\lambda_i}$ (in theory, sample independent)
  - The goal of the factor is to predict the observed covariances among items, so factors represent testable assumptions about the pattern of item covariance
    - Items should be unrelated after controlling for factors → local independence

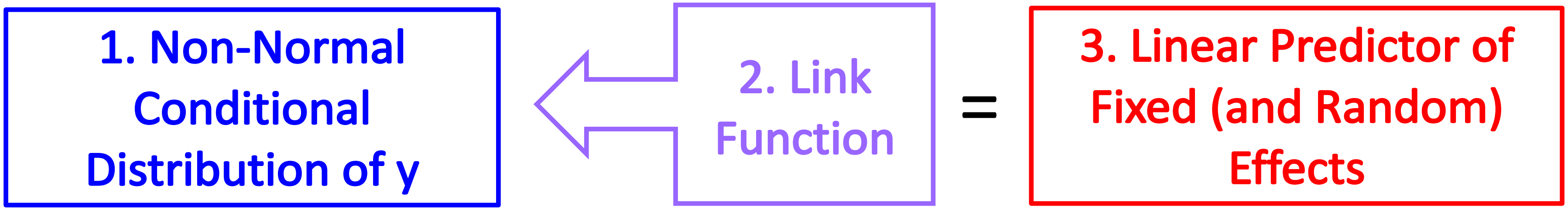
- **Because individual item responses are included:**
  - Items can vary in discrimination (→ Omega reliability) and difficulty
  - To make your test better, you need more BETTER items…
    - With higher standardized factor loadings → with greater information = λ^2/Var(e)
- Measurement error is still assumed constant across the latent trait (one value)

From CFA to IRT and IFA…

Outcome Type → ***Model Family Name***	**Observed Predictor X**	**Latent Predictor X**
Continuous Y → *"General Linear Model"*	Linear Regression	Confirmatory Factor Models
Discrete/categorical Y → *"Generalized Linear Model"*	Logistic/Probit/ Multinomial Regression	Item Response Theory and Item Factor Analysis

- The basis of Item Response Theory (IRT) and Item Factor Analysis (IFA) lies in models for discrete outcomes, which are called "**genera*lized***" linear models
- Thus, IRT and IFA will be easier to understand after reviewing concepts from generalized linear models…

3 Parts of Generalized Linear Models



1. Non-Normal Conditional Distribution of y

2. Link Function

=

3. Linear Predictor of Fixed (and Random) Effects

1. **Non-normal conditional distribution of responses**: how the outcome residuals should be distributed given the sample space (possible values) of the actual outcome

2. **Link Function**: How the conditional mean to be predicted is made **unbounded** so that the model can predict it linearly

3. **Linear Predictor**: How the fixed and random effects of predictors combine additively to predict a link-transformed (continuous) conditional mean

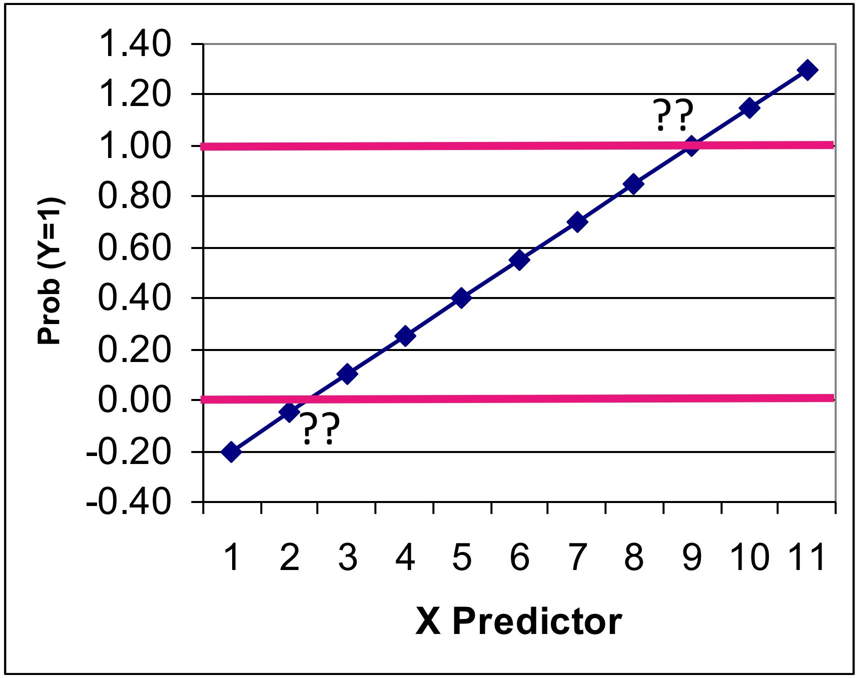
Here's how it works for binary outcomes

- Let's say we have a single binary (0 or 1) outcome…
  - Conditional mean to be predicted for each person is the **probability of having a 1** given the predictors : $\boldsymbol{p}(\mathbf{y_i} = \mathbf{1})$
  - General linear model: $\boldsymbol{p}(\mathbf{y_i} = \mathbf{1}) = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}\mathbf{X_i} + \boldsymbol{\beta_2}\mathbf{Z_i} + \mathbf{e_i}$
    - $\boldsymbol{\beta_0}$ = expected probability when all predictors are 0
    - $\boldsymbol{\beta}$'s = expected change in $\boldsymbol{p}(\mathbf{y_i} = \mathbf{1})$ for a one-unit Δ in predictor
    - $\mathbf{e_i}$ = difference between observed and predicted binary values
  - GLM becomes $\mathbf{y_i} = (\textbf{predicted probability of 1}) + \mathbf{e_i}$
  - **What could possibly go wrong?**

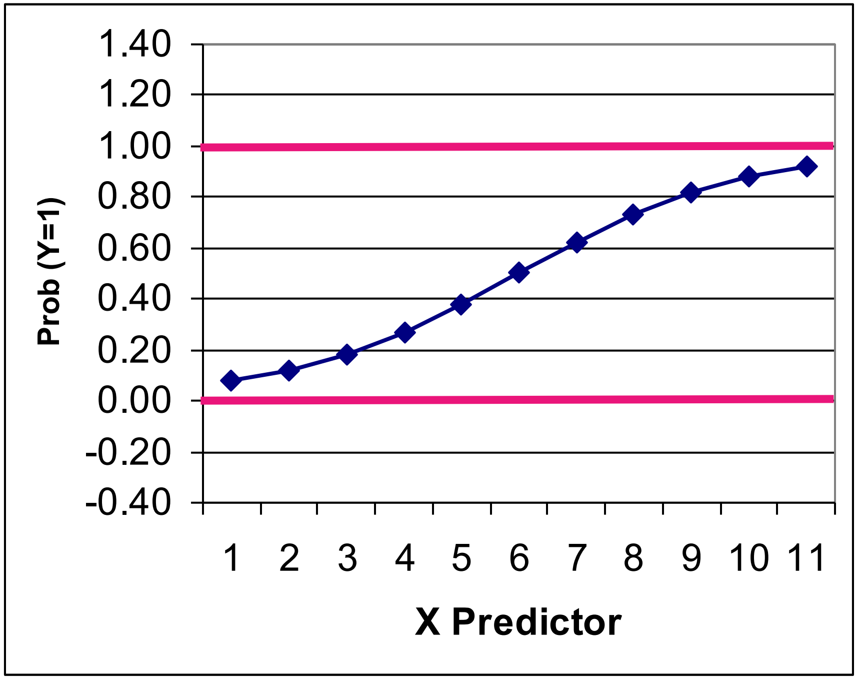
Normal GLM for Binary Outcomes?

- Problem #1: A **linear** relationship between X and Y???
- Probability of a 1 is bounded between 0 and 1, but predicted probabilities from a linear model aren't going to be bounded
- Linear relationship needs to shut off at ends → be nonlinear

We have this…

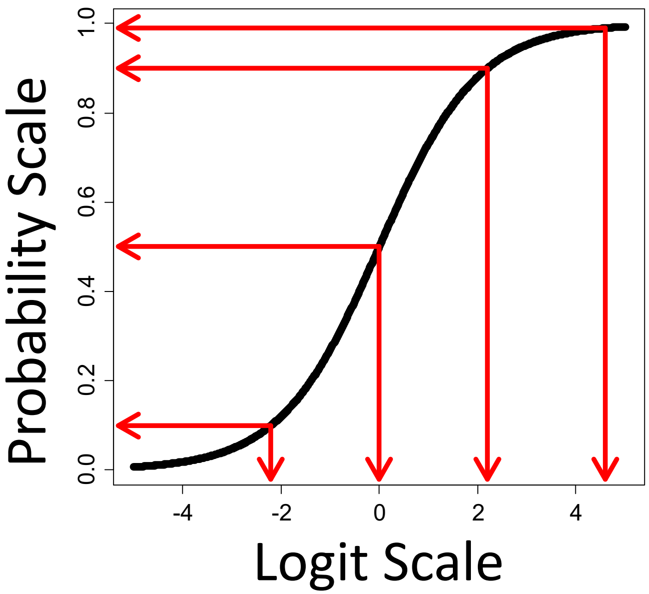


But we need this…



Generalized Models for Binary Outcomes

- Solution to #1: Rather than predicting $\boldsymbol{p}(\mathbf{y_i} = \mathbf{1})$ directly, we must transform it into an unbounded variable with a **link function**:
  - Transform **probability** into an **odds ratio**: $\frac{p_i}{1-p_i} = \frac{\text{prob}(\text{y}_\text{i}=1)}{\text{prob}(\text{y}_\text{i}=0)}$
    - If $p(\text{y}_\text{i} = 1) = .7$ then $\text{Odds}(1) = 2.33$; $\text{Odds}(0) = .429$
    - But odds scale is skewed, asymmetric, and ranges from 0 to +∞ → Not helpful
  - **Take *natural log of odds ratio* → called "logit" link:** $\mathbf{Log}\left[\frac{\boldsymbol{p_i}}{\mathbf{1}-\boldsymbol{p_i}}\right]$
    - If $p(\text{y}_\text{i} = 1) = .7$, then $\text{Logit}(1) = .846$; $\text{Logit}(0) = -.846$
    - Logit scale is now symmetric about 0, range is ±∞ → DING



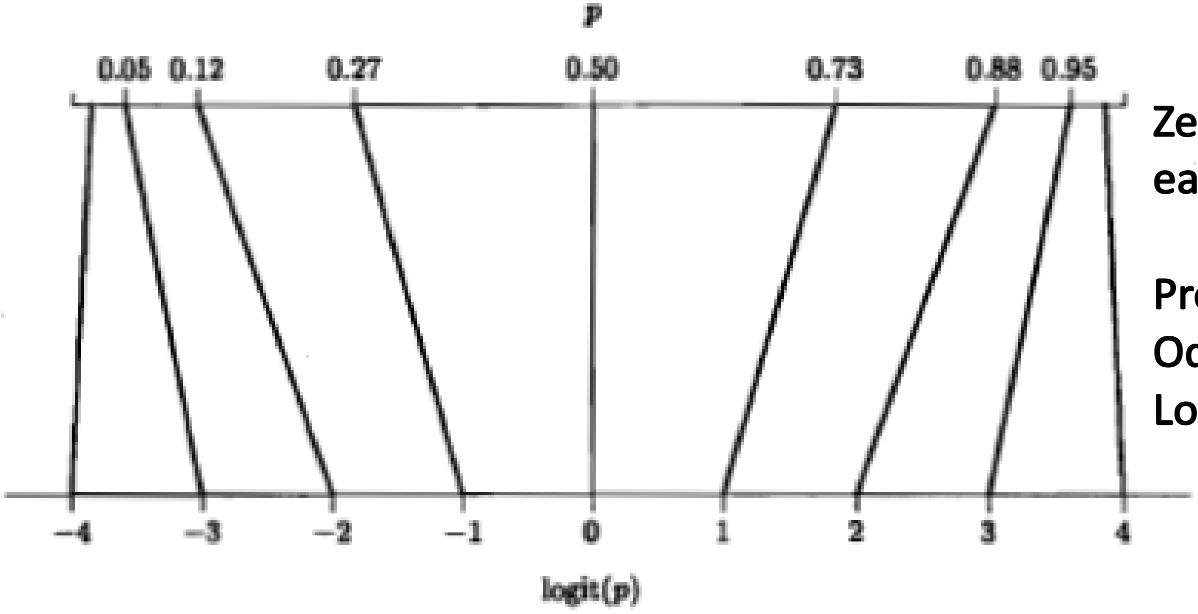
Probability	Logit
0.99	4.6
0.90	2.2
0.50	0.0
0.10	−2.2

Can you guess what $p(.01)$ would be on the logit scale?

Solution to #1: Probability into Logits

- **A Logit link is a <u>nonlinear</u> transformation of probability:**
  - Equal intervals in logits are NOT equal intervals of probability
  - Logits range from ±∞ and are symmetric about prob = .5 (logit = 0)
  - Now we can use a linear model → The model will be **linear with respect to the predicted logit**, which translates into a nonlinear prediction with respect to probability → **the outcome conditional mean shuts off at 0 or 1 as needed**

<u>Probability:</u>
$\boldsymbol{p(\mathrm{y}_{\mathrm{i}} = 1)}$



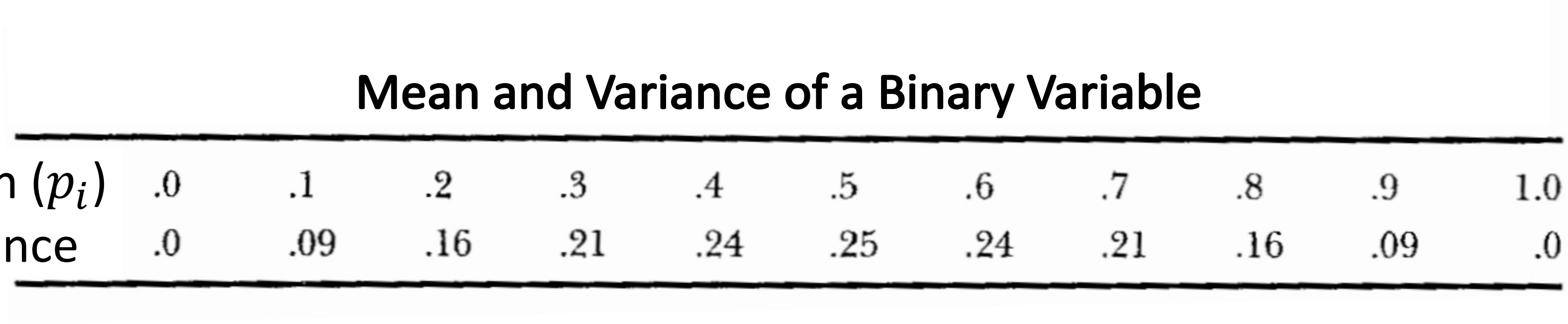
<u>Logit:</u>
$\mathbf{Log}\left[\frac{p_i}{1-p_i}\right]$

Zero-point on each scale:

Prob = .5
Odds = 1
Logit = 0

Normal GLM for Binary Outcomes?

- General linear model: $\boldsymbol{p}(\mathbf{y_i} = \mathbf{1}) = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}\mathbf{X_i} + \boldsymbol{\beta_2}\mathbf{Z_i} + \mathbf{e_i}$
- If $\mathbf{y_i}$ is binary, then $\mathbf{e_i}$ can only be 2 things: $\mathbf{e_i} = \mathbf{y_i} - \hat{\mathbf{y}}_\mathbf{i}$
  - If $\mathbf{y_i} = 0$ then $\mathbf{e_i}$ = (0 − predicted probability)
  - If $\mathbf{y_i} = 1$ then $\mathbf{e_i}$= (1 − predicted probability)
- Problem #2a: So the residuals can't be normally distributed
- Problem #2b: The residual variance can't be constant over X as in GLM because the **mean and variance are dependent**
  - Variance of binary variable: $\mathbf{Var}(\mathbf{y_i}) = \boldsymbol{p_i} * (\mathbf{1} - \boldsymbol{p_i})$



Mean and Variance of a Binary Variable

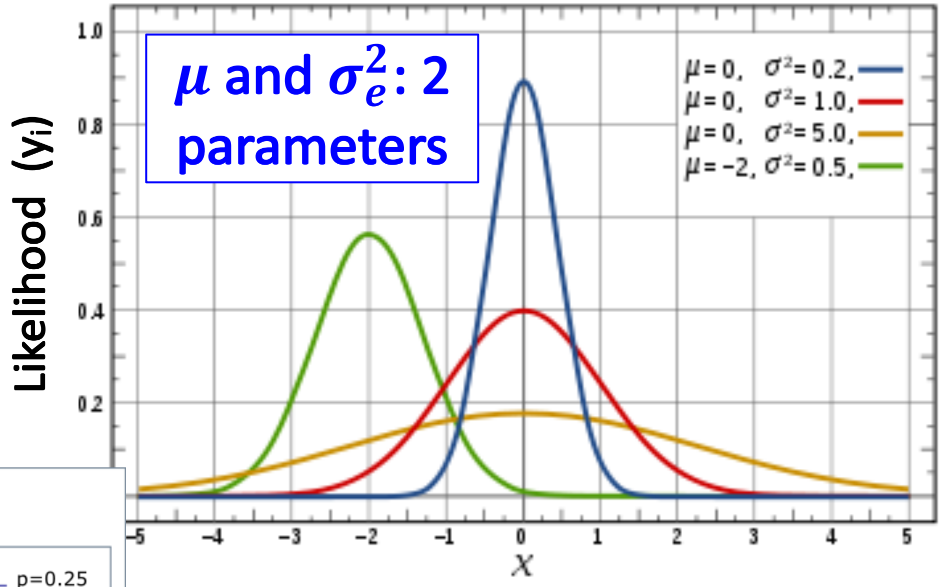
Mean (p_i)	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
Variance	.0	.09	.16	.21	.24	.25	.24	.21	.16	.09	.0

Solution to #2: Bernoulli Distribution

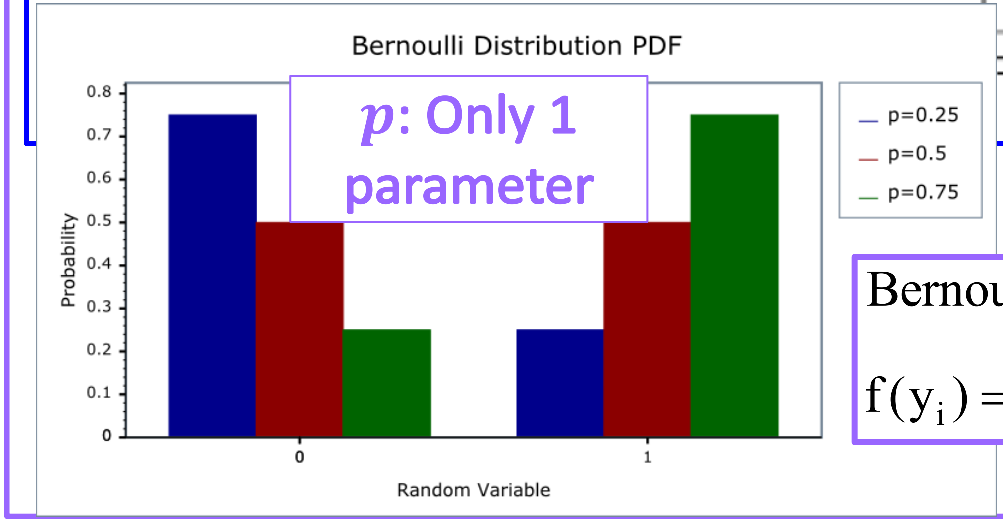
- Instead of a normal residual distribution, we will use a **Bernoulli distribution →** a special case of a binomial for only one outcome

Univariate Normal PDF:

$$f(y_i) = \frac{1}{\sqrt{2\pi\sigma_e^2}} * \exp\left[-\frac{1}{2} * \frac{\left(y_i - \overset{\$}{y}_i\right)^2}{\sigma_e^2} \right]$$



μ and σ_e^2: 2 parameters



p: Only 1 parameter

Bernoulli PDF:

$$f(y_i) = (p_i)^{y_i}(1 - p_i)^{1-y_i}$$

= $p(1)$ if y_i=1, $p(0)$ if y_i=0

Predicted Binary Outcomes

- **Logit:** $\mathbf{Log}\left[\frac{p_i}{1-p_i}\right] = \beta_0 + \beta_1 X_i + \beta_2 Z_i$ ← g(·) link
  - Predictor effects are linear and additive like in GLM, but β = change in **logit(y$_i$)** per one-unit change in predictor

- **Odds:** $\left[\frac{p_i}{1-p_i}\right] = \exp(\beta_0) * (\beta_1 X_i) * (\beta_2 Z_i)$

  or $\left[\frac{p_i}{1-p_i}\right] = \exp(\beta_0 + \beta_1 X_i + \beta_2 Z_i)$

- **Probability:** $p(y_i = 1) = \frac{\exp(\beta_0+\beta_1 X_i+\beta_2 Z_i)}{1+\exp(\beta_0+\beta_1 X_i+\beta_2 Z_i)}$ ← $g^{-1}(\cdot)$ inverse link

  or $p(y_i = 1) = \frac{1}{1+\exp[-1(\beta_0+\beta_1 X_i+\beta_2 Z_i)]}$

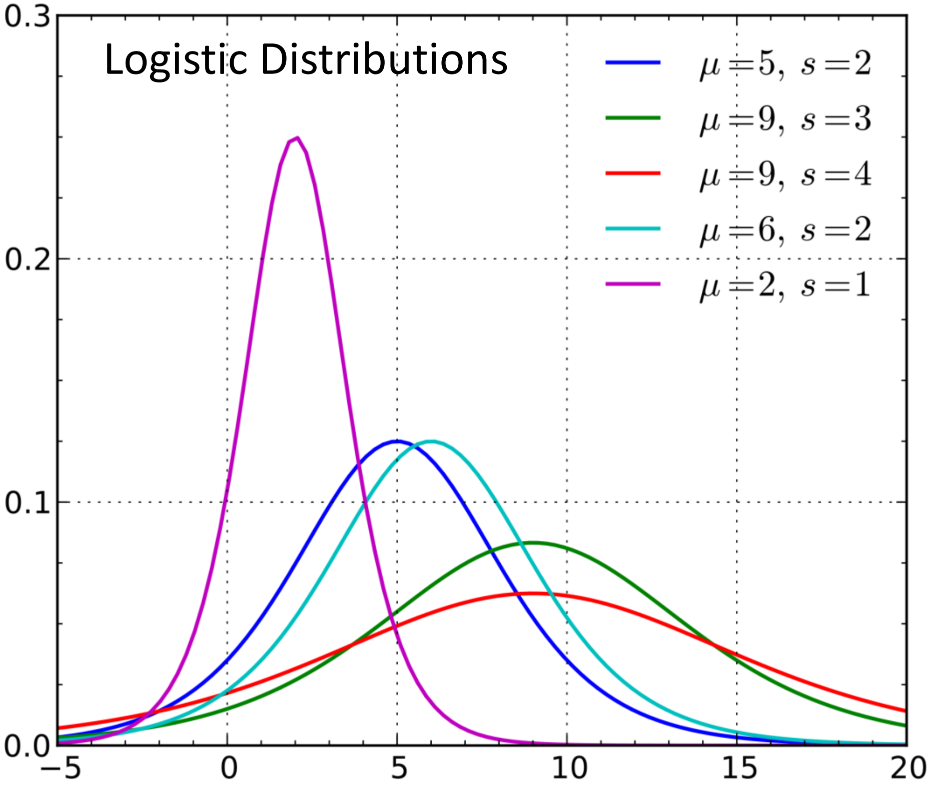
"Latent Responses" for Binary Data

- This model is sometimes expressed by calling the logit(y_i) a underlying continuous ("latent") response of $\mathbf{y_i^*}$ instead:

$$\mathbf{y_i^*} = \boldsymbol{threshold} + \textbf{your model} + \mathbf{e_i}$$

$threshold = \beta_0 * -1$ is given in Mplus, not the intercept

  - In which $\mathbf{y_i} = \mathbf{1}$ if $(y_i^* > threshold)$, or $\mathbf{y_i} = \mathbf{0}$ if $(y_i^* \le threshold)$



So **if predicting $\mathbf{y_i^*}$**, then

$$e_i \sim \text{Logistic}(0, \sigma_e^2 = 3.29)$$

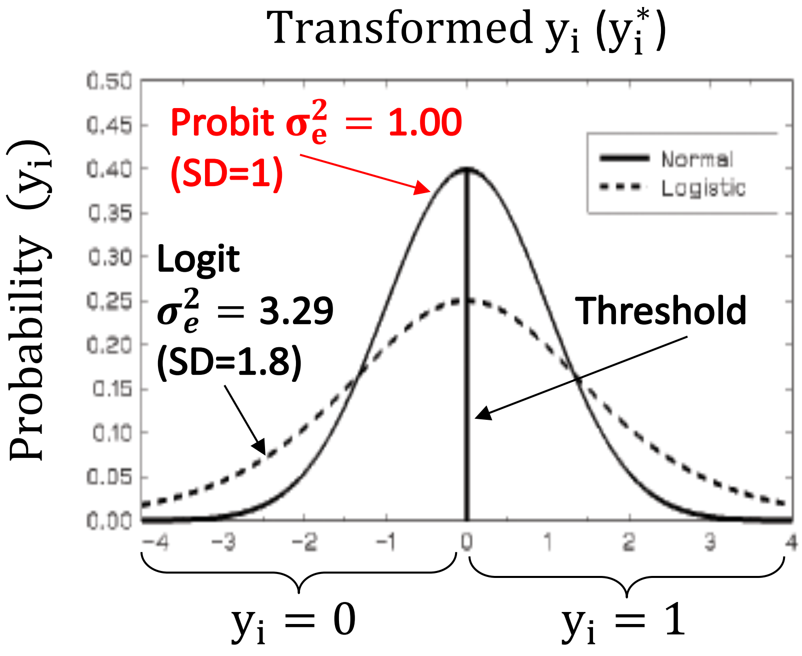
Logistic Distribution:

Mean = μ, Variance = $\frac{\pi^2}{3}s^2$, where *s* = scale factor that allows for "over-dispersion" (must be fixed to 1 for binary responses for identification)

Other Models for Binary Data

- The idea that a "latent" continuous variable underlies an observed binary response also appears in a **Probit Regression** model:
  - A ***probit*** link, such that now your model predicts a different transformed $\mathrm{y_i}$:
    $$\mathrm{Probit}(\mathrm{y_i} = 1) = \Phi^{-1} p(\mathrm{y_i} = 1) = \mathit{your\ model} \xleftarrow{\quad} \boxed{\mathbf{g}(\cdot)}$$
    - Where $\boldsymbol{\Phi}$ = standard normal cumulative distribution function, so the transformed $\mathrm{y_i}$ is the **z-score** that corresponds to the value of standard normal curve below which observed probability is found (requires integration to transform back)
  - Same Bernoulli distribution for the binary $\mathrm{e_i}$ residuals, in which residual variance cannot be separately estimated (so no $\mathrm{e_i}$ in the model)
    - Probit also predicts "latent" response: $\mathrm{y_i^*} = \mathrm{threshold} + \ \mathrm{your\ model} + \mathrm{e_i}$
    - But Probit says $\mathrm{e_i} \sim \mathrm{Normal}(0, \sigma_e^2 = 1.00)$, whereas Logit $\sigma_e^2 = \frac{\pi^2}{3} = 3.29$
  - So given this difference in variance, probit estimates are on a different scale than logit estimates, and so their estimates won't match… however…

Probit vs. Logit: Should you care? No


Transformed y_i (y_i^*)

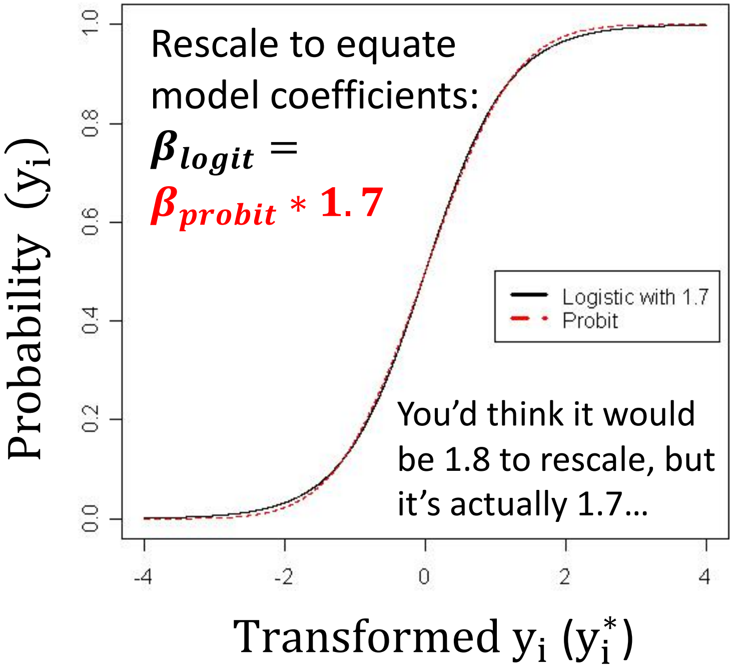
Probit $\sigma_e^2 = 1.00$ (SD=1)

Logit $\sigma_e^2 = 3.29$ (SD=1.8)

Threshold

$y_i = 0$

$y_i = 1$


Rescale to equate model coefficients: $\beta_{logit} = \beta_{probit} * 1.7$

You'd think it would be 1.8 to rescale, but it's actually 1.7…

- Other fun facts about probit:
  - Probit = "ogive" in the Item Response Theory (IRT) world
  - Probit has no odds ratios (because it's not based on odds)

- Both logit and probit assume **symmetry** of the probability curve, but there are other *asymmetric* options as well…

How IRT/IFA are the same as CFA

- **NOW BACK TO YOUR REGULARLY SCHEDULED MEASUREMENT CLASS**
- **IRT/IFA** = measurement model in which latent trait estimates depend on both persons' responses and items' properties
  - Like CFA, **both items and persons matter**, and thus properties of both are included in the measurement model
  - Items differ in sample-independent difficulty and discrimination as in CFA →These are represented by translatable quantities in IRT and IFA
- After controlling for a person's latent trait score (now called **Theta**), the item responses should be uncorrelated (also called local independence)
  - The ONLY reason item responses are correlated is a unidimensional Theta
  - If this is unreasonable, we can fit multidimensional factor models instead, and then responses are independent after controlling for ALL Thetas
  - Can be violated by other types unaccounted for multidimensionality or dependency (e.g., method factors, common stem "testlets")

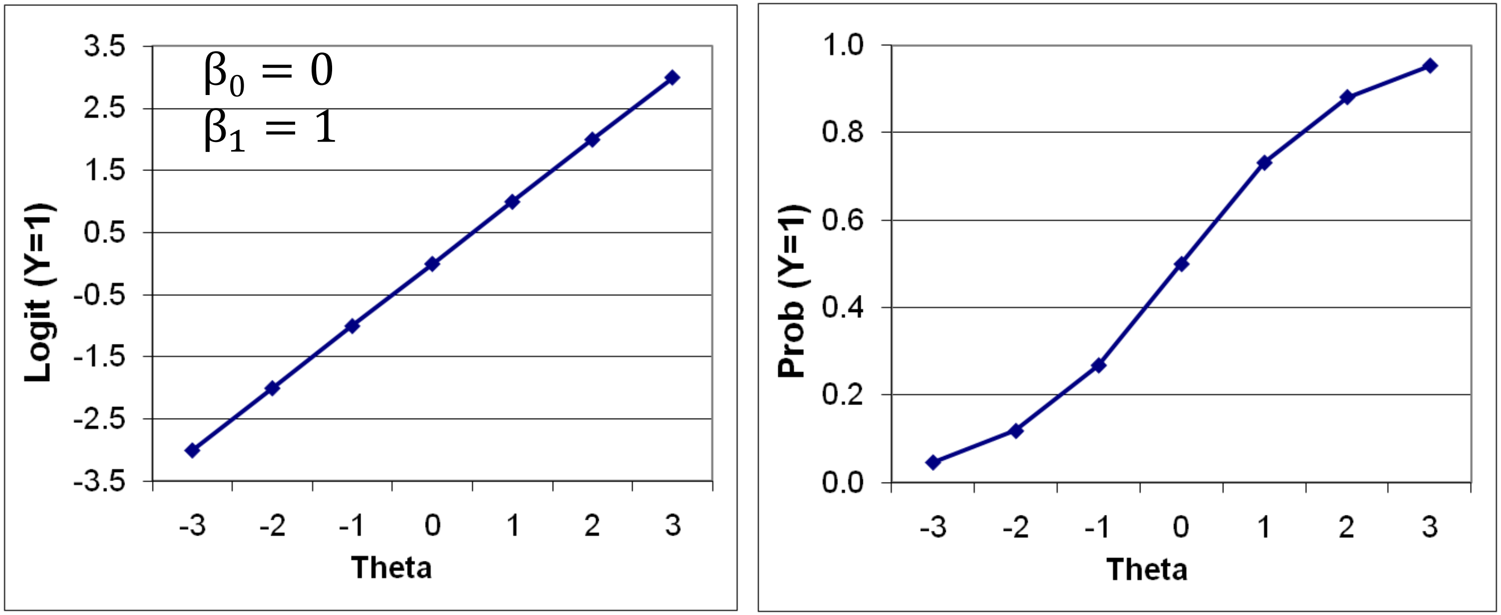
How IRT/IFA are *different from* CFA

- IRT/IFA uses the same family of **link functions** (transformations) as in generalized models, it's just that the predictor isn't measured directly
  - IRT/IFA = logistic regression instead of linear regression
  - Predictor = Latent factor/trait in IRT/IFA = "Theta" and its slopes are still supposed to predict the covariance across item responses, just like in CFA

- **IRT/IFA specifies a <u>nonlinear</u> relationship between binary, ordinal, or nominal item responses and the latent trait (now called "Theta")**
  - Probability is bounded between 0 and 1, so the effect (slope) of Theta must be nonlinear, so it will shut off at the extremes of Theta (S-shaped curve)
  - Errors cannot have constant variance across Theta or be normally distributed
  - Full information models use logit (σ^2_{e*}= 3.29) or probit (σ^2_{e*}= 1.00) link functions, but limited information models only have probit (σ^2_{e*} = 1.00)
    - Logit = 1.7*Probit, so it's pretty much the same result either way
    - Probit in IRT models is called "ogive" (as discussed in Embretson & Reise)

Nonlinearity in IRT/IFA

- The relationship between Theta and the probability of response=1 is **“nonlinear”** → **a monotonic s-shaped logistic curve** whose shape and location are dictated by the estimated item parameters
  - **Linear** prediction of the **logit**, **nonlinear** prediction of **probability**



- It may be that other kinds of non-linear relationships could be more appropriate and thus fit better → These are “non-parametric” IRT models

Item Response Theory (IRT) "vs" Item Factor Analysis (IFA) Models

Mplus can do ALL of these model/estimator combinations:	Model form with **discrimination** and **difficulty** parameters	Model form with **loadings** and **thresholds**
Full Information via Maximum Likelihood ("Marginal ML") → *uses <u>original</u> item responses*	**"IRT"**	**"?"**
Limited Information via Weighted Least Squares ("WLSMV") → *uses item response <u>summary</u>*	**"?"**	**"IFA"**

- CFA assumes normally distributed, continuous item responses, but CFA-like models also exist for categorical responses → these are **IRT** and **IFA**
- These different names are used to reflect the combination of how the model is specified and how it is estimated, but it's the same core model

Model Format in IRT and IFA

- Item Factor Analysis (IFA) models look very similar to CFA, but Item Response Theory (IRT) models look very different
- Partly due to predicting logits/probits (IFA) vs. probability (IRT):
  - **Logit:** $\mathbf{Log}\left[\frac{p_i}{1-p_i}\right] = \beta_0 + \beta_1 X_i + \beta_2 Z_i$
  - **Probability:** $p(y_i = 1) = \frac{\exp(\beta_0+\beta_1 X_i+\beta_2 Z_i)}{1+\exp(\beta_0+\beta_1 X_i+\beta_2 Z_i)}$
- Partly due to different model formats (stay tuned)
- These two model forms are just re-arrangements of each other, but historically have been estimated using different methods (full vs. limited information) and for different purposes
- Mplus provides both kinds of output for binary data, but only IFA output for categorical data (we will calculate IRT version)
- We'll start with IRT for binary responses, then move to IFA …

Simplest IRT Model: One-Parameter Logistic (1-PL or Rasch) Model for Binary Responses (0/1)

- 1PL model is written in different, but equivalent ways (Embretson & Reise):
  - **Logit**: $$\mathrm{Log}\left(\frac{p\left(y_{is}=1\right)}{1-p\left(y_{is}=1\right)}\,|\,\theta_s\right)=\theta_s-b_i$$
  - **Probability**: $$P\left(y_{is}=1\,|\,\theta_s\right)=\frac{\exp\left[\theta_s-b_i\right]}{1+\exp\left[\theta_s-b_i\right]}$$
  - **θ_s = subject ability** → most likely latent trait score (called Theta) for subject *s* given the pattern of item responses
  - **b_i = item difficulty** → location on latent trait (like an intercept, but it's actually 'difficulty' now!)
- Probability of response=1 depends on person ability (theta) vs. item difficulty:
  - If ability > difficulty, then logit > 0, probability > .50
  - If difficulty > ability, then logit < 0, probability < .50

Fundamentals of IRT

- **Back in CTT**, scores only have meaning relative to the persons in the same sample, and thus **sample norms** are needed to interpret a person's score
    - "I got a 12. Is that good?"
      "Well, that puts you into the 90th percentile."
      "Great!"
    - "I got a 12. Is that good?"
      "Well, that puts you into the 10th percentile."
      "Doh!"
    - Same score in both cases, but different reference group!

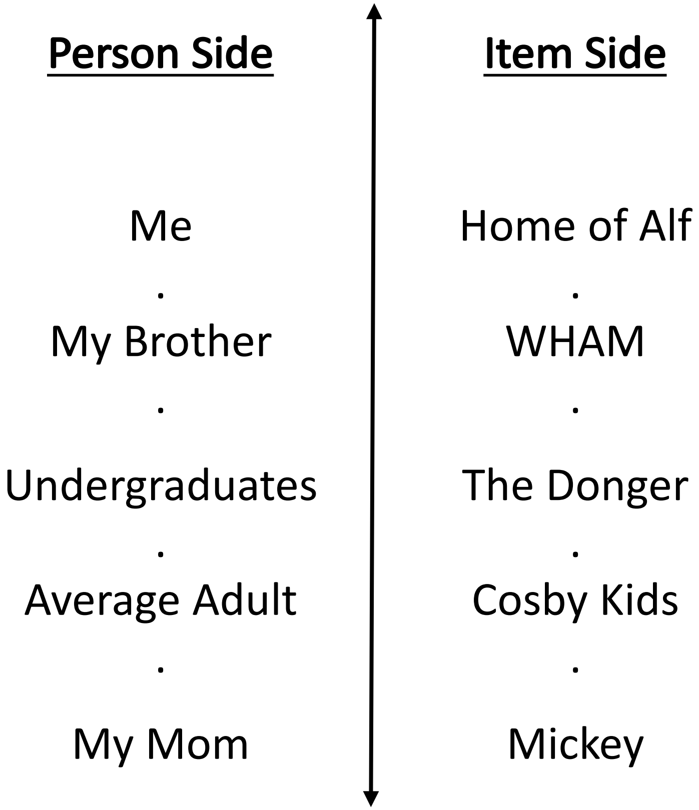
- **In IRT**, the properties of items and persons are placed along the same underlying latent continuum= "**conjoint scaling**"
    - This concept can be illustrated using **construct maps** that order both persons in terms of ability and items in terms of difficulty…

A Construct Map Example

Theta θ_s = Item difficulty level at which one has a 50% probability of response=1

A Latent Continuum of 80s Pop Culture Knowledge

Theta θ_s is interpreted relative to item locations, not group norms



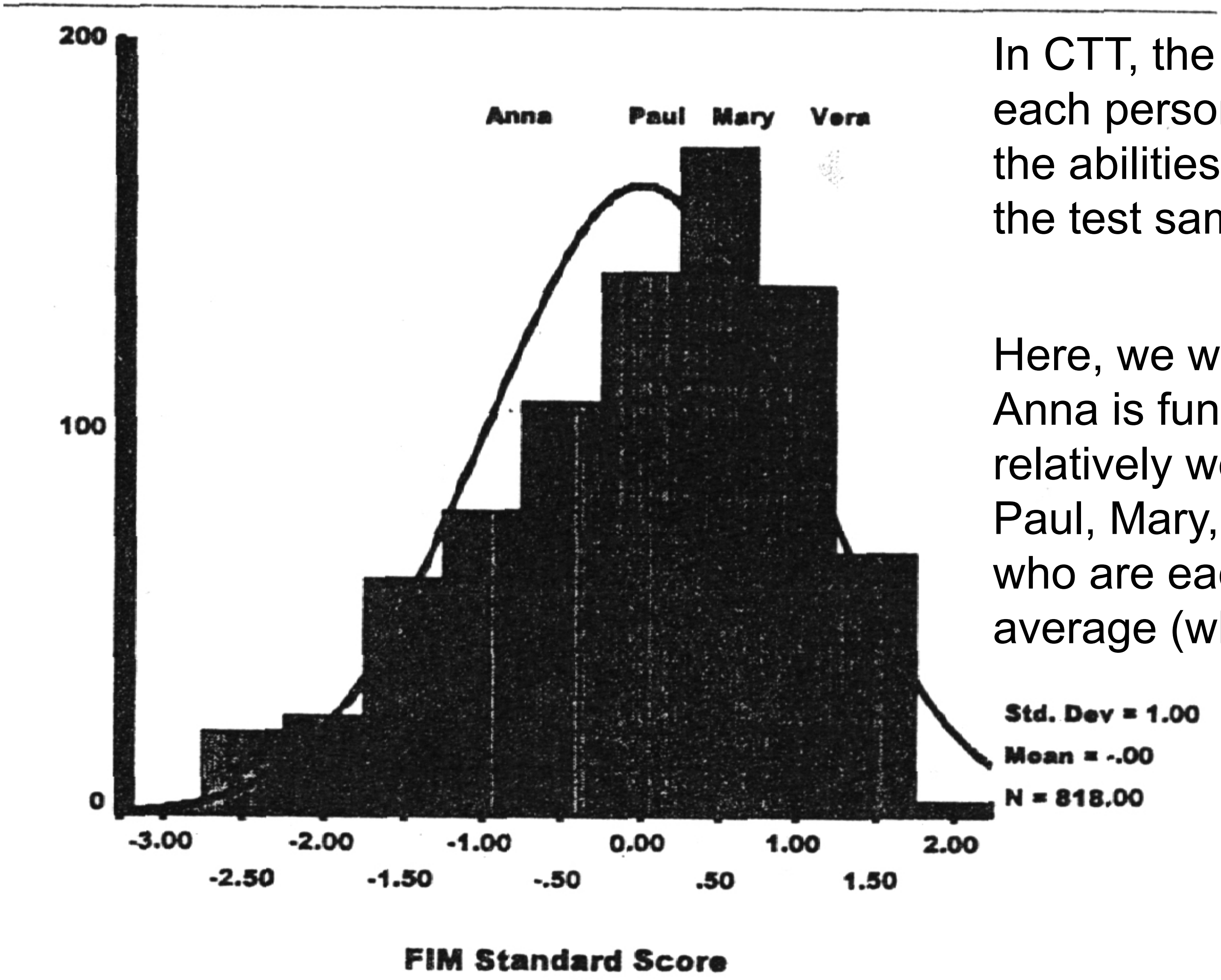
Person Side	Item Side
Me	Home of Alf
.	.
My Brother	WHAM
.	.
Undergraduates	The Donger
.	.
Average Adult	Cosby Kids
.	.
My Mom	Mickey

Persons are ordered by Theta ability/severity

Items are located in order by difficulty/severity

Person Theta and Item Difficulty share the same latent metric

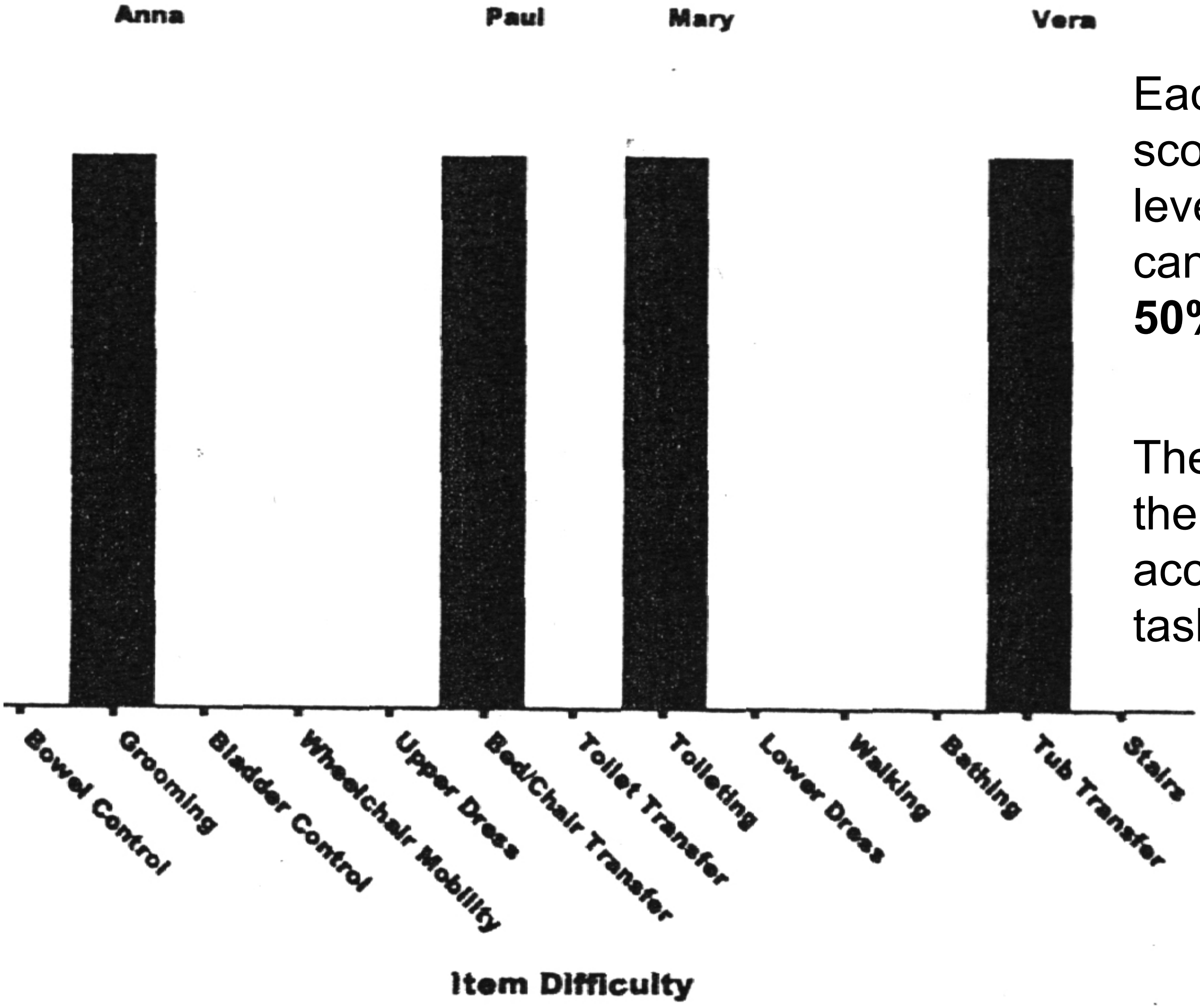
Norm-Referenced Measurement in CTT



In CTT, the ability level of each person is relative to the abilities of the rest of the test sample.

Here, we would say that Anna is functioning relatively worse than Paul, Mary, and Vera, who are each above average (which is 0).

Item-Referenced Measurement in IRT



Each person's Theta score reflects the level of activity they can do on their own **50% of the time**.

The model predicts the probability of accomplishing each task given Theta.

Interpretation of Theta

- **Theta estimates are 'sample-free' and 'scale-free'**
  - Theta estimate does not depend on who took the test with you
  - Theta estimate does not depend on which items were on the test
    - After calibrating all items to the same metric, can get a person's location on latent ability metric regardless of which *particular* items were given

- However: although the Theta estimate does not depend on the particular items, its ***standard error*** does
  - Extreme Thetas without many items of comparable difficulty will not be estimated that well → large SE (flat likelihood)
  - Likewise, items of extreme difficulty without many persons of comparable ability will not be estimated that well → large SE

Another version of the 1PL (Rasch) Model

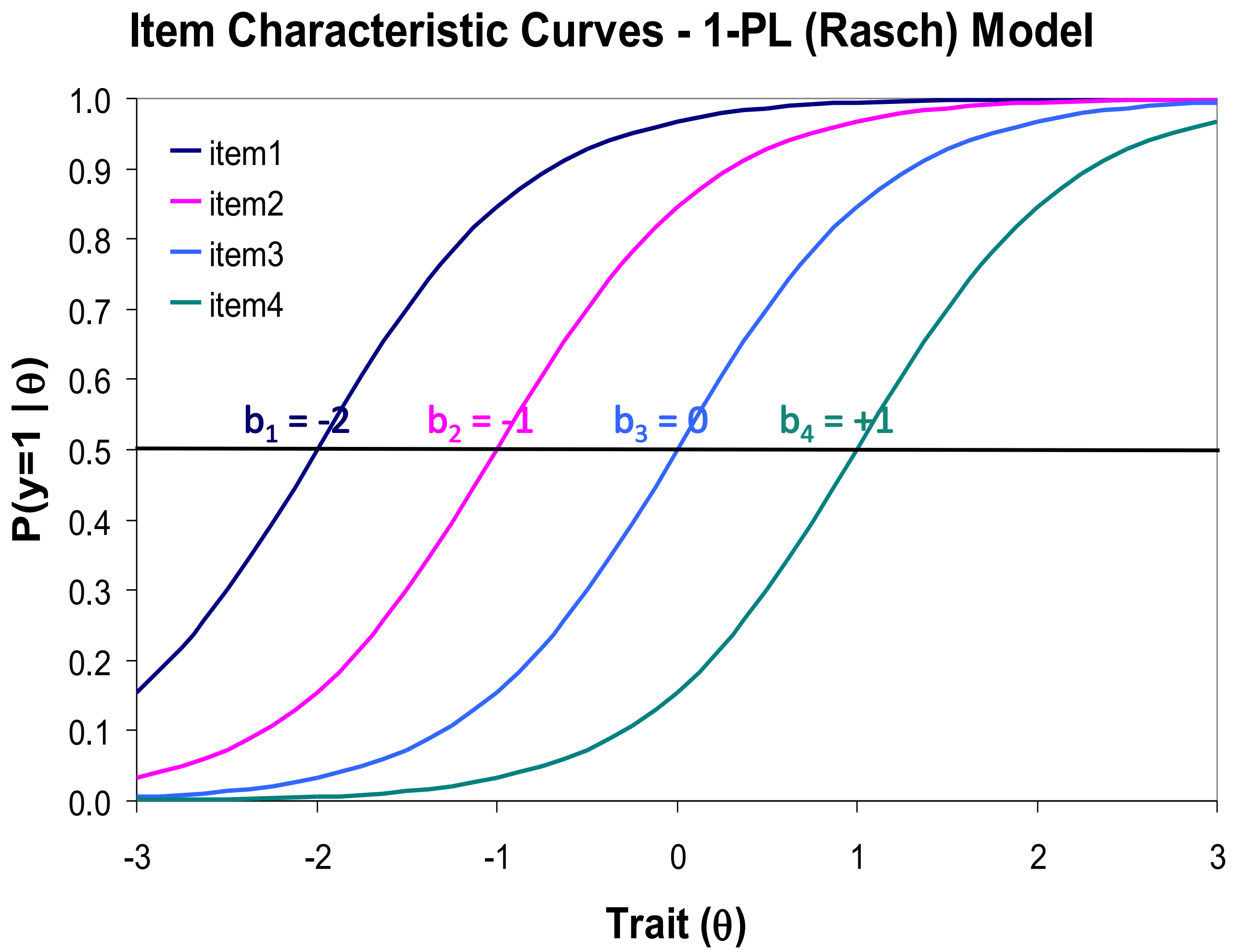
- **Logit:** $$\text{Log}\left(\frac{p(y_{is}=1)}{1-p(y_{is}=1)}\,|\,\theta_s\right)=a(\theta_s-b_i)$$

- **Probability:** $$P(y_{si}=1\,|\,\theta_s)=\frac{\exp[a(\theta_s-b_i)]}{1+\exp[a(\theta_s-b_i)]}$$

Parameter from a **probit** (ogive) model will be smaller by a factor of 1.7

- **a = “discrimination”** = relation of item to latent trait = slope of the s-shape curve as it crosses probability = .50 (its max slope)
- You’ll note that the 1-PL model has “$\mathbf{a}$” and not “$\mathbf{a_i}$” – that’s because $\mathbf{a}$ is assumed constant across items (and thus, the 1 parameter that is estimated for each item is still difficulty $\mathbf{b_i}$)
- If using the probit link function, the predicted outcome is the z-score for the area to the left under the normal curve for that predicted probability
- Previously Mplus factored out $\mathbf{1.7}$ next to the $\mathbf{a}$ so that the model parameters would be comparable regardless of using a probit or logit link, but the 1.7 is now embedded in the parameters instead

1-PL (Rasch) Model Predictions



b_i = item difficulty location on latent trait at which probability = .50

a = discrimination slope at prob = .50, (logit = 0, which is point of inflection)

Note: **equal a terms** means curves will never cross → this idea is called "Specific Objectivity"

Can you guess what's next?
2-Parameter Logistic Model (2PL)

- The 1-PL (Rasch) model assumes tau-equivalence → equal discrimination

- Thus, the 2-PL frees this constraint by changing "$\mathbf{a}$" to "$\mathbf{a_i}$":
  - **Logit:** $$\mathrm{Log}\left(\frac{p(y_{is}=1)}{1-p(y_{is}=1)}\,|\,\theta_s\right)=a_i\left(\theta_s-b_i\right)$$
  - **Probability:** $$P(y_{si}=1\,|\,\theta_s)=\frac{\exp[a_i(\theta_s-b_i)]}{1+\exp[a_i(\theta_s-b_i)]}$$
  - $\mathbf{a_i}$ **= "discrimination"** = relation of **each item** to latent trait = slope of the s-shape curve when it crossed probability = .50 (its max slope)
  - $\mathbf{b_i}$ is still difficulty (location where probability = .50)
  - Note that $\mathbf{a_i}$ is a **linear** slope for theta θ predicting the **logit of $\mathbf{y_{is}}$=1** but a **nonlinear** slope for theta θ predicting the **probability of $\mathbf{y_{is}}$=1**

Parameter from a **probit** (ogive) model will be smaller by a factor of 1.7

“IRT” vs. “Rasch”

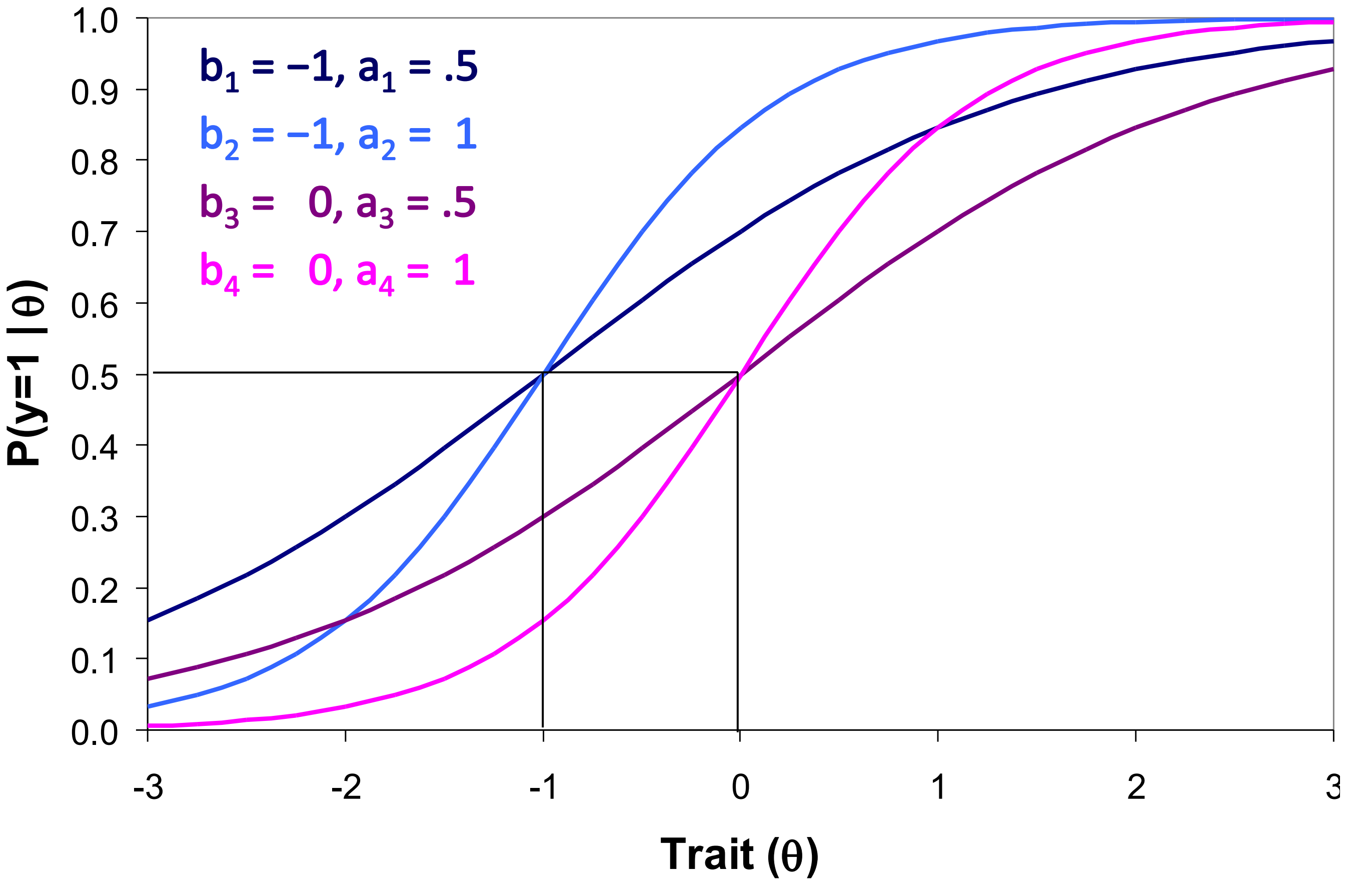
- According to most **IRT** people, a “Rasch” model is just an IRT model with item discrimination $\mathbf{a_i}$ held equal across items (a tau-equivalent model)
  - Rasch = 1-PL where $\mathbf{b_i}$ item difficulty is the only item parameter
  - Slope = discrimination $\mathbf{a_i}$ = strength of relation of item to latent trait (theta θ)
  - *“Items may not be equally ‘good’, so why not just let their slopes vary?”*

- According to most **Rasch** people, the 2PL and rest of IRT is voo-doo
  - Rasch models have specific properties that are lost once you allow the item curves to cross (by using **item-varying $\mathbf{a_i}$**) → “**Loss of Specific Objectivity**”
    - Under the Rasch model, persons are ordered the same in terms of predicted responses regardless of which item difficulty location you’re looking at
    - Under the Rasch model, items are ordered the same in terms of predicted responses regardless of what level of person theta you’re looking at
    - The $\mathbf{a_i}$ **represents a theta*item interaction** → the item curves cross, so the ordering of persons or items is no longer invariant, and this is “bad”
  - *“Items should not vary in discrimination if you know your construct.”*

Item Characteristic Curves: 2PL Model

$\mathbf{b_i}$ **= difficulty** = location on latent trait at which p = .50 (or logit = 0)

$\mathbf{a_i}$ **= discrimination** slope at p = .50 (at the point of inflection of curve)



Note: **unequal a_i** → curves cross → violates Specific Objectivity

At Theta θ = −1: Items 3 and 4 are 'harder' than 1 and 2 (lower prob=1)

At Theta θ = +2: Item 1 is now 'harder' than Item 4 (lower prob=1)

"IRT" vs. "Rasch": What Goes into Theta

- In Rasch models, **the sum score is a 'sufficient statistic'** for Theta
  - For example, given 5 items ordered in difficulty from easiest to hardest, each of these response patterns where **3/5 are correct** would yield the **same estimate of Theta**:

    1 1 1 0 0 (most consistent)
    0 1 1 1 0
    0 0 1 1 1
    1 0 1 0 1 (???)
    …. (and so forth)

- In 2-parameter models, items with higher discrimination (a_i) **count more** towards Theta (and SE will be lower for tests with higher a_i items)
  - It not only matters **how many** items you got correct, but **which ones**
  - Rasch people don't like this idea, because then ordering of persons on Theta is dependent on the item properties

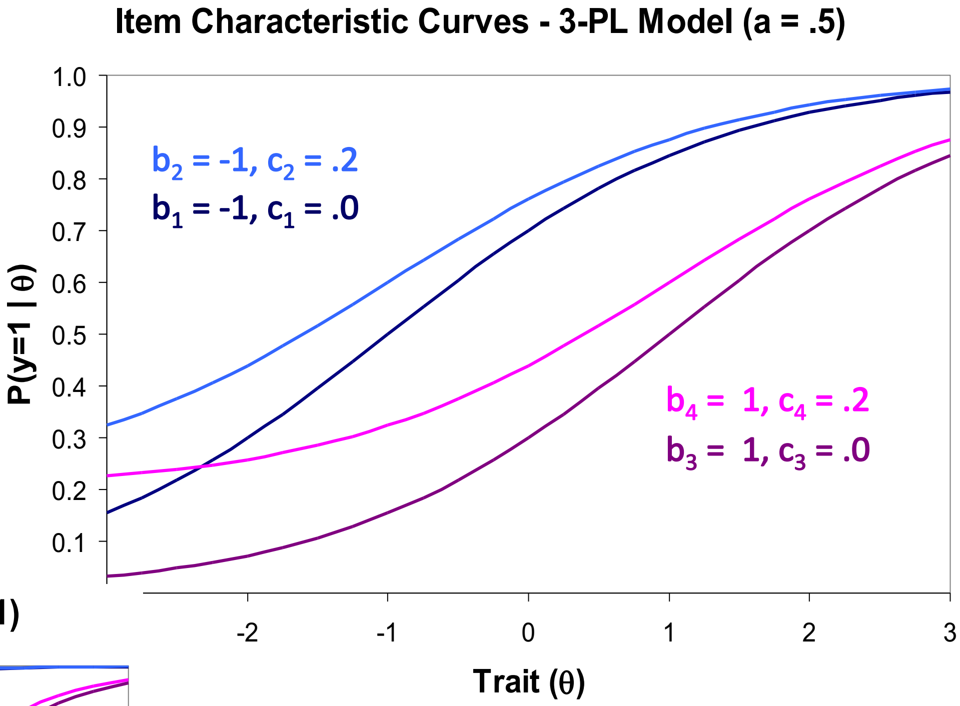
Yet Another Model for Binary Responses: 3-Parameter Logistic Model (3PL)

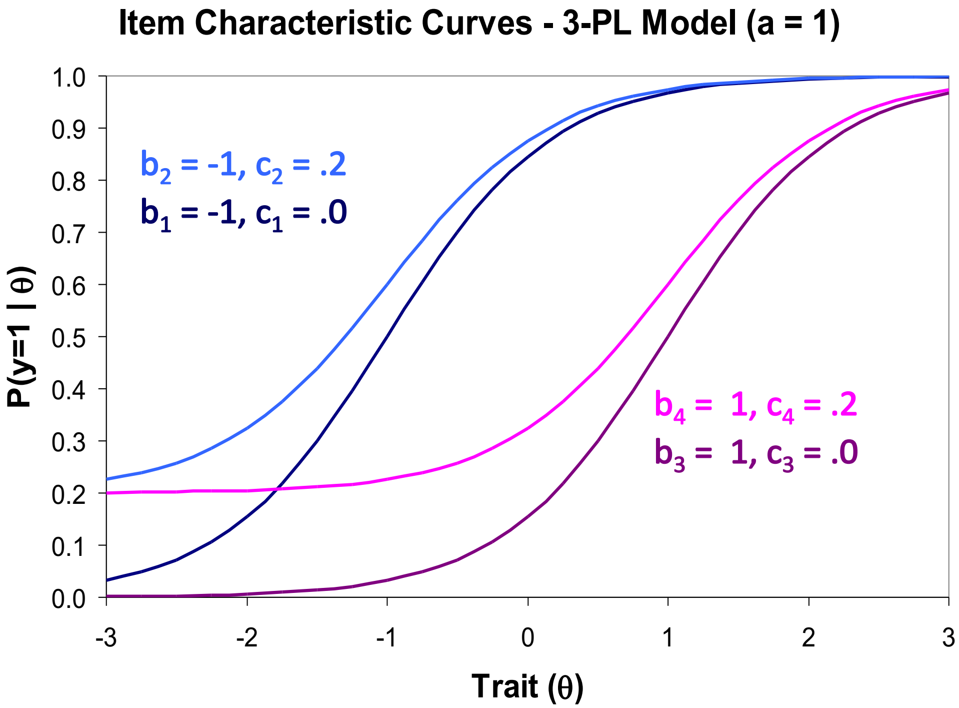
$$\text{Probability}(y_{si}=1|\theta_s)=\boxed{c_i+(1-c_i)}\frac{\exp[a_i(\theta_s-b_i)]}{1+\exp[a_i(\theta_s-b_i)]}$$

- $\mathbf{b_i}$ = item difficulty → location
  - Higher values → more difficult items (lower chance of a 1)
- $\mathbf{a_i}$ = item discrimination → slope
  - Higher values = more discriminating items = better items
- $\mathbf{c_i}$ = item lower asymptote → "guessing" (where $\mathbf{c_i} > \mathbf{0}$)
  - Lower bound of probability independent of Theta
  - e.g., would be around .25 given 4 equally guess-able multiple choice responses
  - Could estimate a common c across items as an alternative
- Probability starts at guessing c_i then depends on Theta θ and a_i, b_i
  - 3-PL model is now available within Mplus 7.4; c_i is labeled as $2
  - Require 1000s of people because c_i parameters are hard to estimate—you have to have enough low theta people to determine what the probability of guessing is likely to be

Top: Items with lower discrimination ($\mathbf{a_i = .5}$)

Below: Items with higher discrimination ($\mathbf{a_i = 1}$)





Note that item difficulty b_i values are no longer where prob = .50 → the expected prob at b_i is increased by the lower asymptote c_i parameter

One Last Model for Binary Responses: 4-Parameter Logistic Model (4PL)

$$\text{Probability}(y_{si}=1|\theta_s)=c_i+\boxed{(d_i-c_i)}\frac{\exp[a_i(\theta_s-b_i)]}{1+\exp[a_i(\theta_s-b_i)]}$$

- $\mathbf{b_i}$ = item difficulty → location
- $\mathbf{a_i}$ = item discrimination → slope
- $\mathbf{c_i}$ = item lower asymptote → “guessing”
- $\mathbf{d_i}$ = item upper asymptote → “**carelessness**” (so $d_i < 1$)
  - Maximum probability to be achieved independent of Theta
  - Could be carelessness or unwillingness to endorse the item no matter what

- Probability starts at ‘guessing’ c_i tops out at ‘carelessness’ d_i then depends on Theta θ and a_i, b_i in between
  - 4-PL model is now available within Mplus 7.4; c_i and d_i are labeled as $2 and $3
  - But good luck estimating it! May need to use a common c and d instead

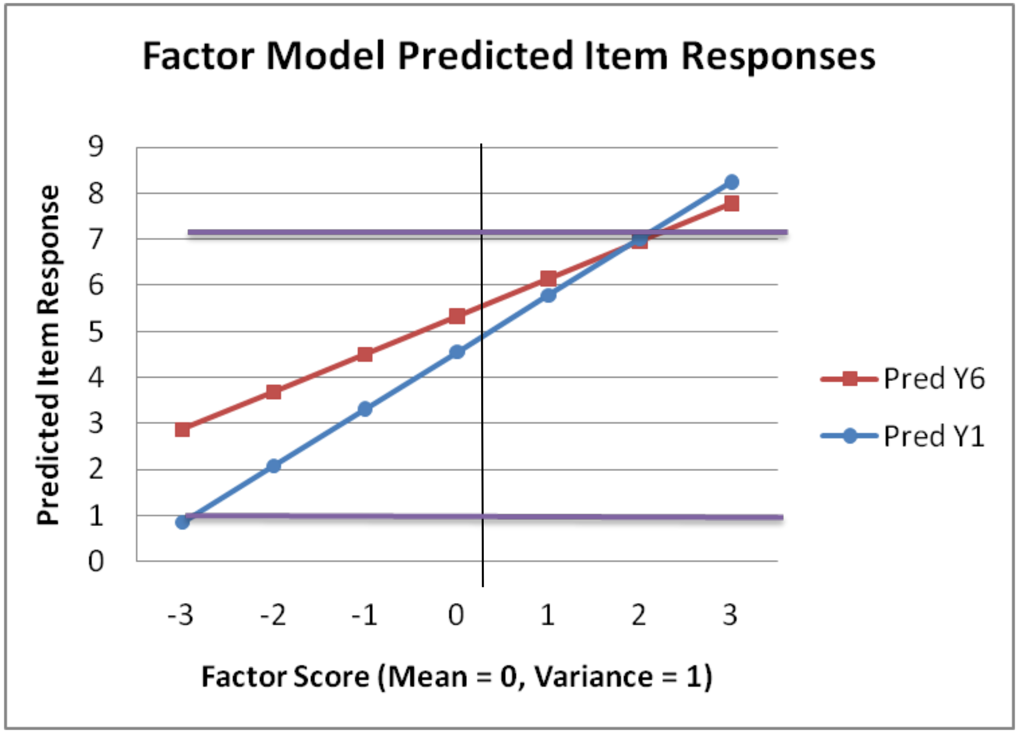
Anchoring: Model Identification in IRT

- As in CFA, we have a latent trait (a pretend variable) without a scale, so we need to give Theta θ a scale (it needs a mean and a variance)
  - This is called "**anchoring**" in IRT → CFA calls it "**model identification**"
  - There are two equivalent options: Anchor by Persons or Anchor by Items
- **Anchor by persons**: Fix Theta θ mean = 0 and Theta θ variance = 1
  - This is the "Z-score" approach to model identification used in CFA
  - All item difficulties b_i and item discriminations a_i are then estimated
- **Anchor by items**: Fix one item difficulty b_i = 0 and one item a_i = 1
  - "Marker item" approach to model identification
  - Mean and variance of Theta θ are estimated instead
  - Fixing mean of item difficulty = 0 is another way (more common in Europe)
- Big picture: as in CFA, the numerical scale doesn't matter, all that matters is that persons and items are on the same scale → "conjoint scaling"

Information: Reliability in IRT Models

- "**Information**" ≈ reliability → measurement precision
- In **CFA models** (continuous Y), item-specific "information" is rarely referred to, but it is easy to compute:
  - How good is my item → how much information is in it?
    - How much of its variance is "true" (shared with the factor) relative to how much of its variance is "error"?
    - **Information = unstandardized loading^2 / error variance**
  - Note that information will be **constant** across trait level in CFA
    - Items with a greater proportion of true variance are better, the end
    - "Information function" is FLAT across ability level
  - How do I make my test better?
    - **More items with more information** (with stronger factor loadings)
  - Sum of information across items = **Test information function**
    - Test information function will also be flat across trait level in CFA

Item Information in CFA Models



$y_6 = 5.32 + 0.82(F_s) + e_6$
$e_6{}^2 = 1.67$

$y_1 = 4.55 + 1.23(F_s) + e_1$
$e_1{}^2 = 1.53$

Info $y_6 = 0.82^2 / 1.67 = .401$

Info $y_1 = 1.23^2 / 1.57 = .998$

Std $y_6 = 3.48 + 0.54(F_s) + e_6$

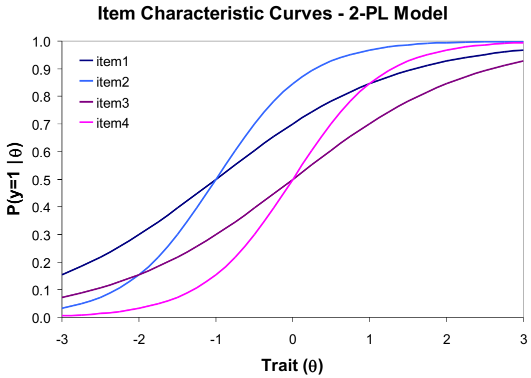
Std $y_1 = 2.60 + 0.71(F_s) + e_1$

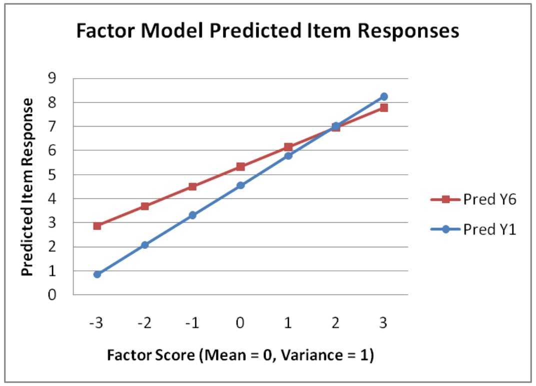
- CFA has a **linear slope (factor loading)** → predicts the same increase in the y_{is} item response for a one-unit change in F (all across levels of F)
- y_1 **has more information** than y_6 (and a higher standardized factor loading), so y_1 is better than y_6, period (for all possible factor scores)

Test Information in IRT Models

- Test information can be converted to reliability as follows:
  - **Reliability = information / (information+1)**
    - **Information of 4 converts to reliability of .80**
    - **information of 9 converts to reliability of .90**
- This formula comes from classical test theory:
  - Reliability = true var / (true var + error var)
  - Reliability = 1 / (1 + error var), where error var = 1/info
  - Reliability = 1 / 1 + (1/info) → info / (info+1)
- An analog of overall model-based reliability (omega) could be computed by summing reliabilities for each possible theta, weighted by the number of people at each level of Theta, but that's missing the point…
- Because the slopes relating Theta to the probability of an item response are non-linear, this means that **reliability must VARY over Theta**

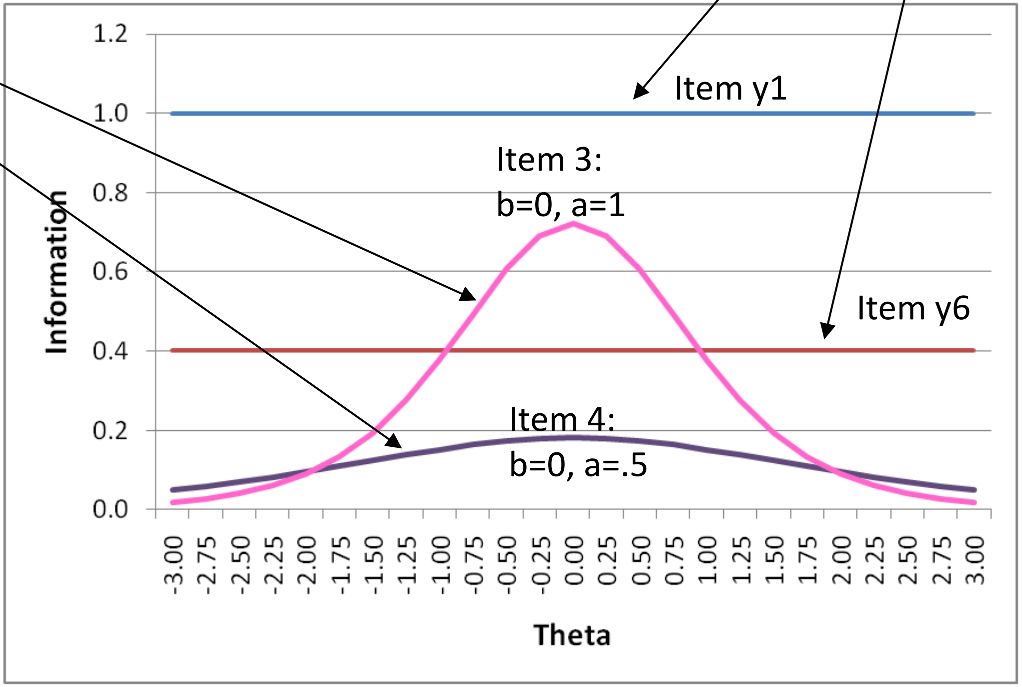
Item Information in CFA vs. IRT





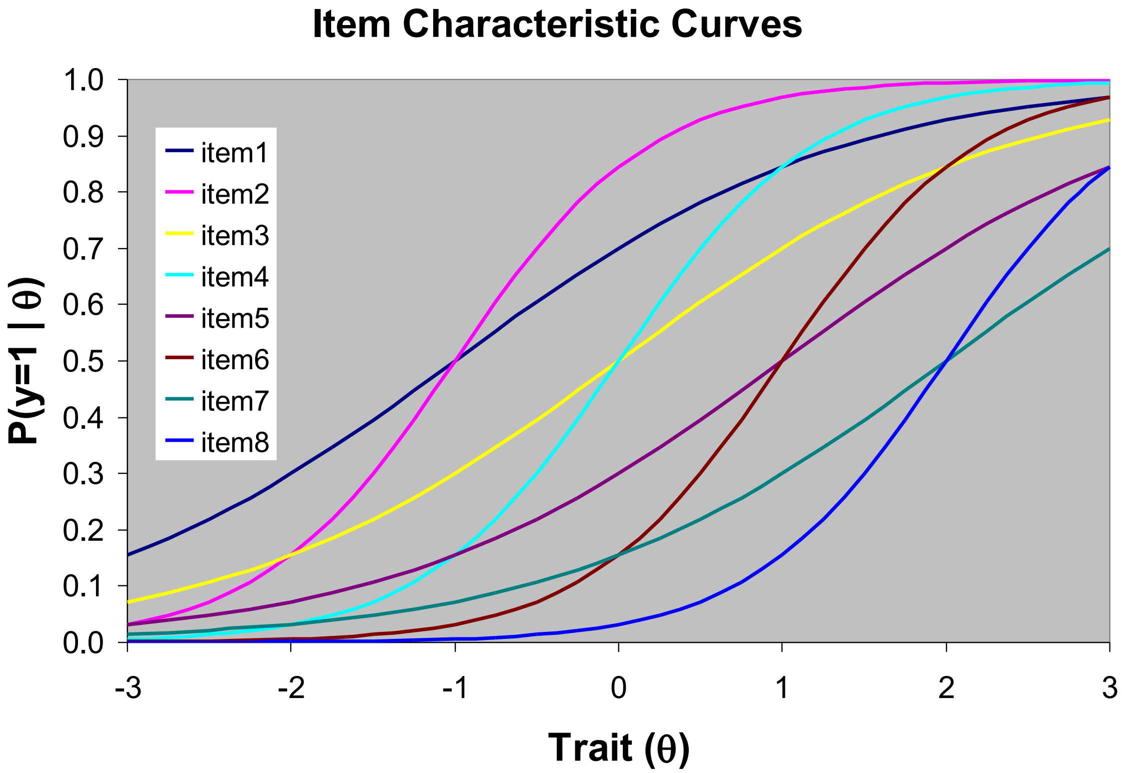
CFA Item Information Functions

IRT Item Information Functions



Effects of Item Parameters on Item Characteristic Curves

Item	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
a discrimination	0.5	**1.0**	0.5	**1.0**	0.5	**1.0**	0.5	**1.0**
b difficulty	-1.0	-1.0	0.0	0.0	1.0	1.0	2.0	2.0

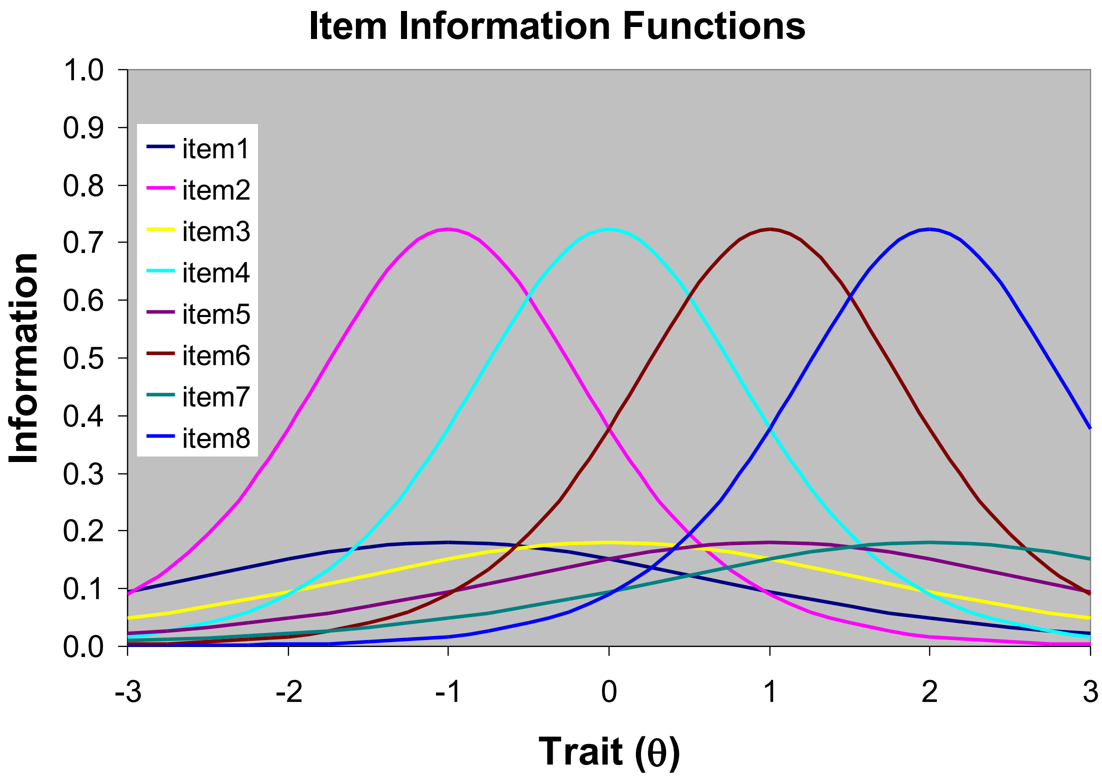


An important result of the non-linear slopes in an IRT model is that the **slope stops working** (so reliability decreases) as you move away from the item difficulty location.

In the **CFA** model with linear slopes, **the slope never stops working** (at least in theory).

Effects of Item Parameters on Item Information Curves

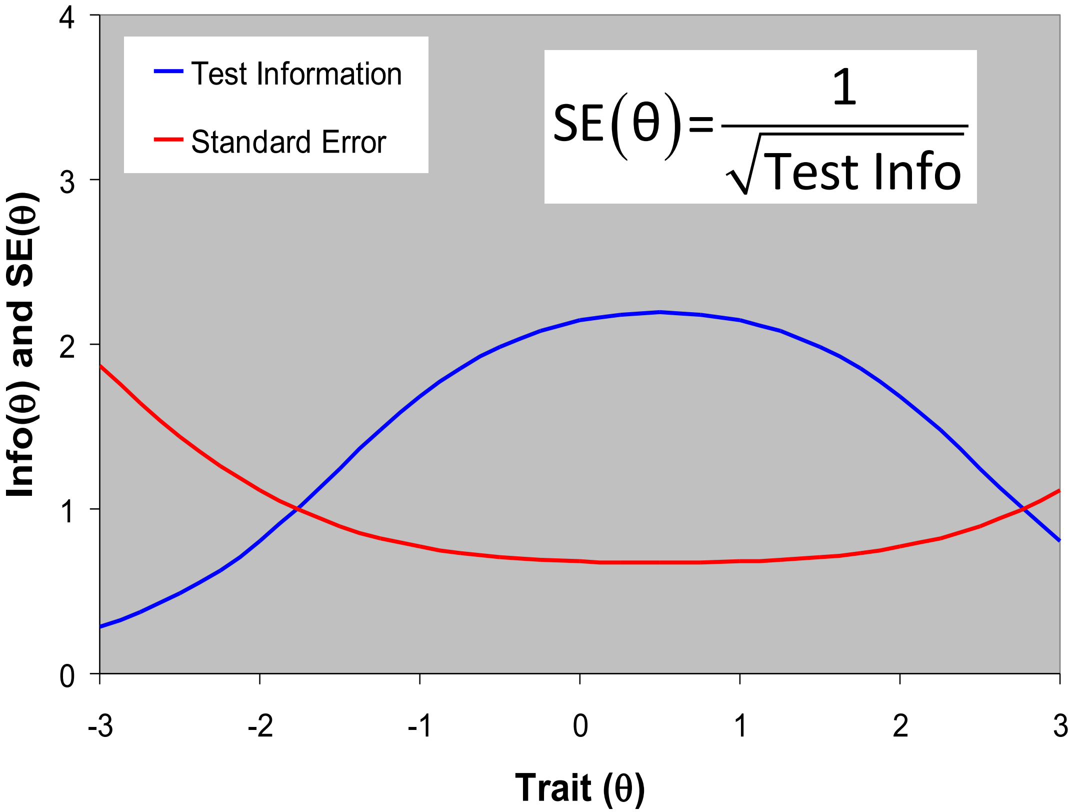
Item	1	2	3	4	5	6	7	8
a discrimination	0.5	**1.0**	0.5	**1.0**	0.5	**1.0**	0.5	**1.0**
b difficulty	-1.0	-1.0	0.0	0.0	1.0	1.0	2.0	2.0



Information (reliability) is maximized around the item difficulty location.

Items with greater a_i item discrimination values have greater absolute information.

Test Information (and SE) by Theta



$$SE(\theta)=\frac{1}{\sqrt{\text{Test Info}}}$$

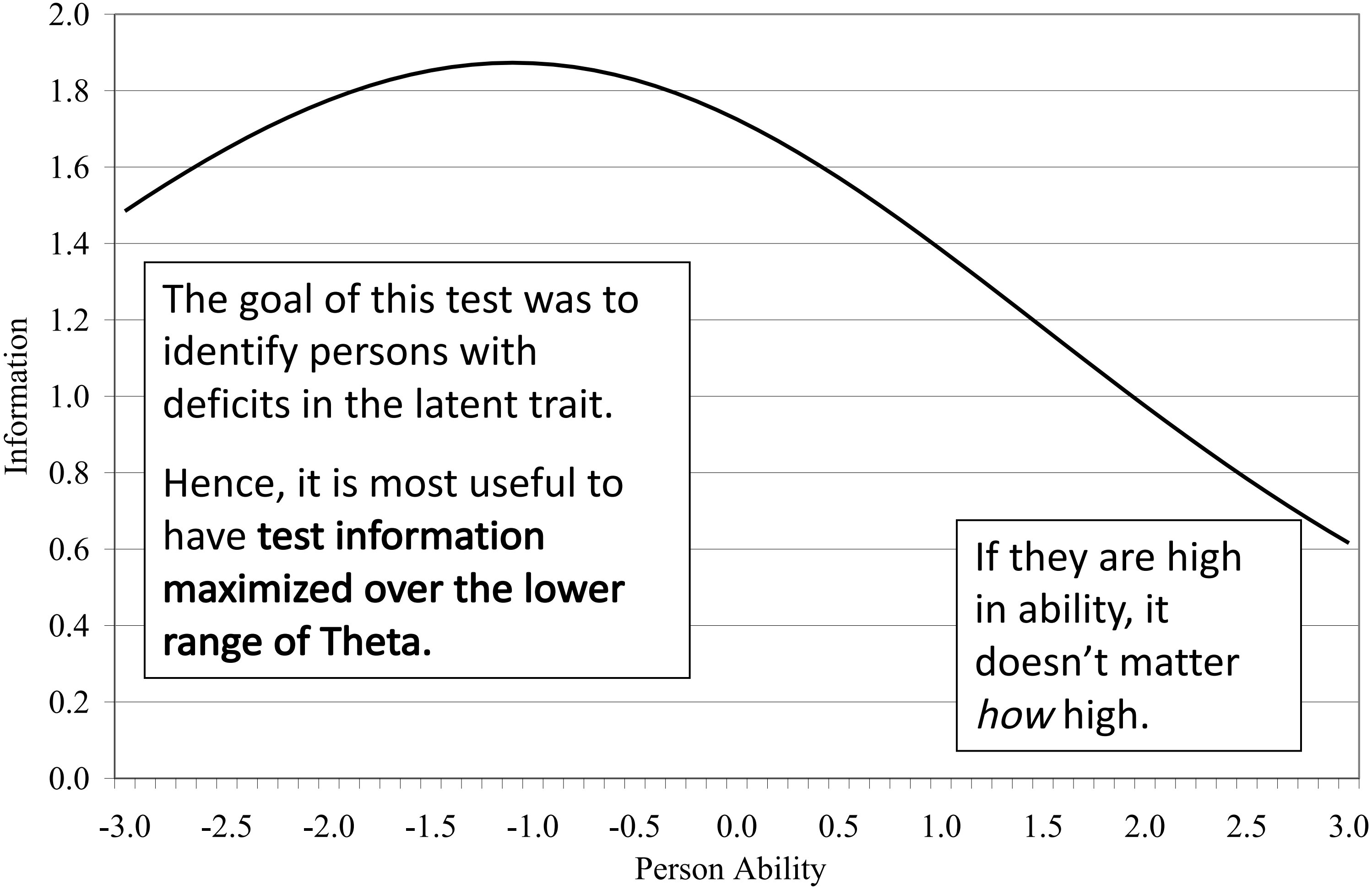
If you sum all the item information curves, you get a **test information** curve that describes how reliable your test is over the range of Theta.

Test Information is very useful to know—it can tell you where the 'holes' are in your measurement precision, and guide you in adding/removing items.

There is no single 'ideal' test information function—only what is optimal for **your** purposes of measurement. Here are a few examples….

Another Example of (Not-So-Good) Test Information

But test info only gets up to ~2…



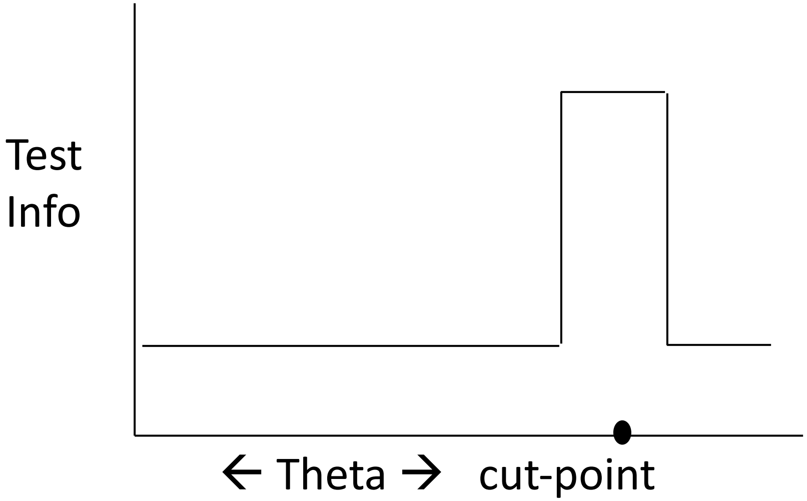
The goal of this test was to identify persons with deficits in the latent trait.

Hence, it is most useful to have **test information maximized over the lower range of Theta.**

If they are high in ability, it doesn't matter *how* high.

Other Shapes of Test Information

- If the goal is to measure a trait across persons equally well, and you expect people to be normally distributed, then your best bet is to create a test with information highest in the middle (where most people are likely to be)
- If your goal is to identify individuals below or above a cut-point, however, your **test information function** should ideally look more like this:
  - You'd want to **maximize sensitivity around the cut-point region,** and otherwise not waste time measuring people well who are nowhere near the cut-point
  - If **classifying people** is the goal of measurement, however, you might be better off with a different family of latent trait models in which Theta is already categorical instead → "**Diagnostic Classification Models**"



How to Improve Your Test

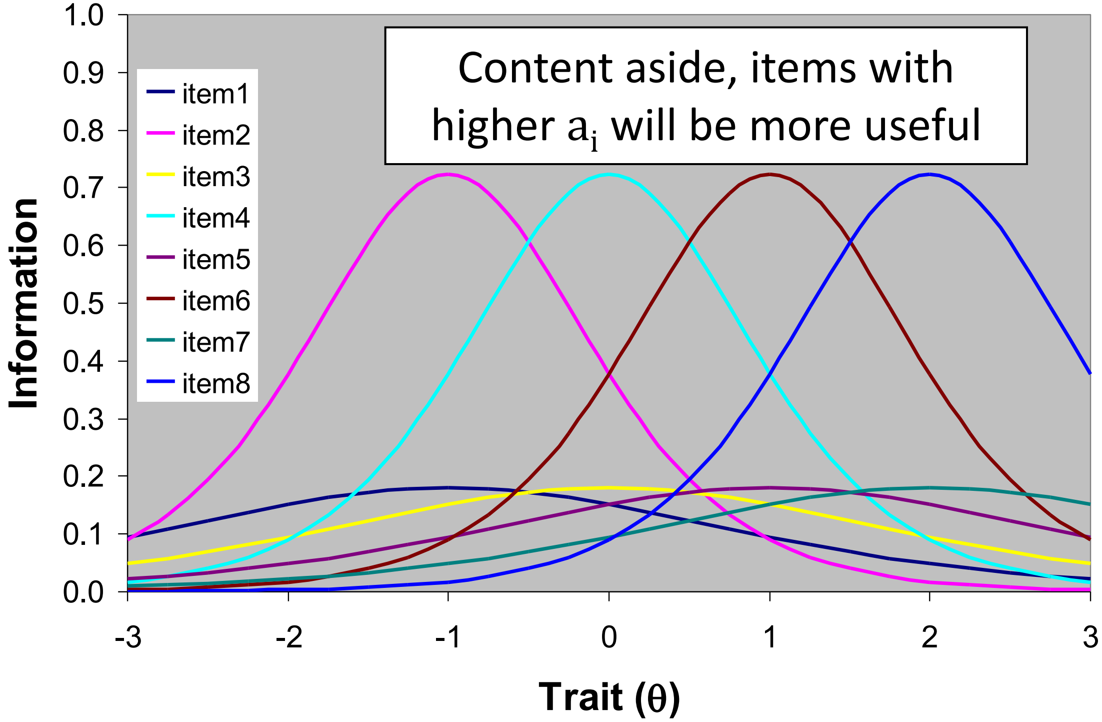
- In CTT, because item properties are not part of the model, items are seen as exchangeable, and more items is better
  - Thus, *any* new item is *equally* better for the model

- In CFA and IRT, more items is still better…
  - **In CFA, the question is "how much better"?**
    - This depends on the standardized loading; intercepts are not important
    - Specifies a linear relationship between theta and the item responses, so 'for whom' isn't relevant—a better item is better for everyone equally
  - **In IRT, the question is "how much better, and for whom?"**
    - This depends on the discrimination ($\mathrm{a_i}$ slope) and the difficulty ($\mathrm{b_i}$ location), respectively (difficulties are important, and are always estimated)
    - Because of the nonlinear relationship between theta and the item responses, items are only useful for thetas in the middle of their S-curves

Effects of Item Parameters on Item Information Curves

Item	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
a discrimination	0.5	**1.0**	0.5	**1.0**	0.5	**1.0**	0.5	**1.0**
b difficulty	-1.0	-1.0	0.0	0.0	1.0	1.0	2.0	2.0

Item Information Functions



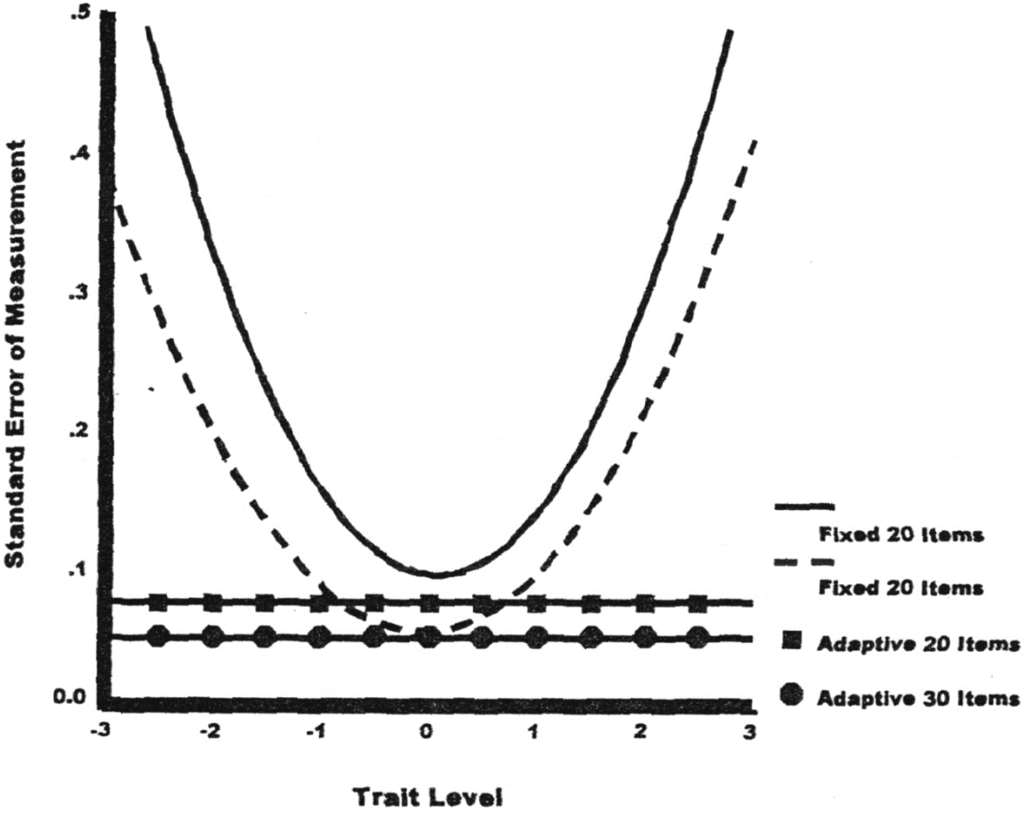
Content aside, items with higher a_i will be more useful

In addition to a_i item discrimination, though, you want to make sure you are covering the range of difficulty where you want to measure people best.

IRT and Adaptive Testing: *Fewer Items Can Actually Be Better*

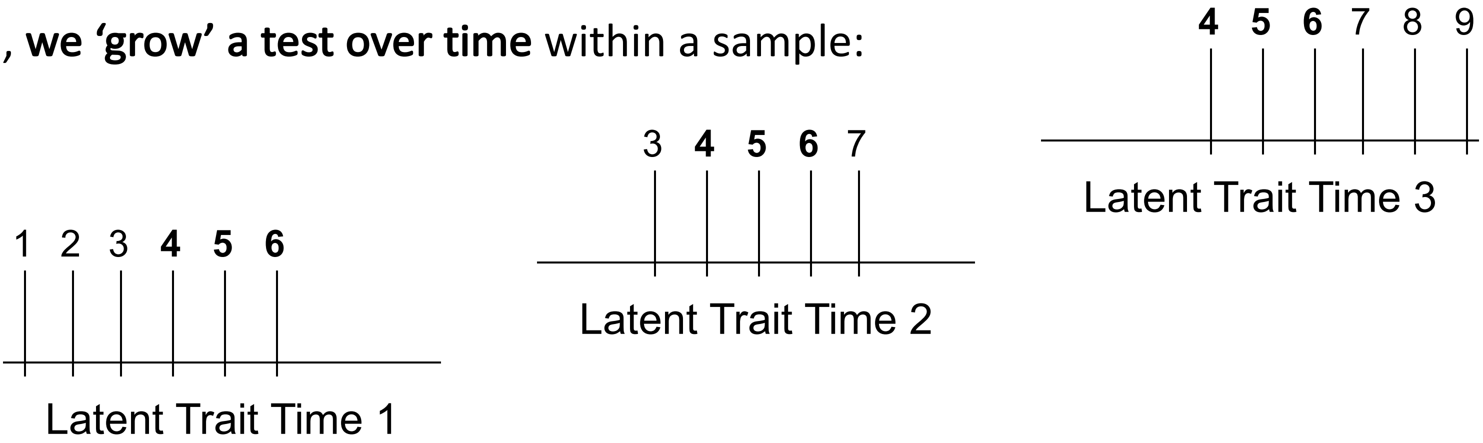
- In a normal distribution of the latent trait and a comparable distribution of item difficulty, **extreme people are usually measured less well** (higher SE).

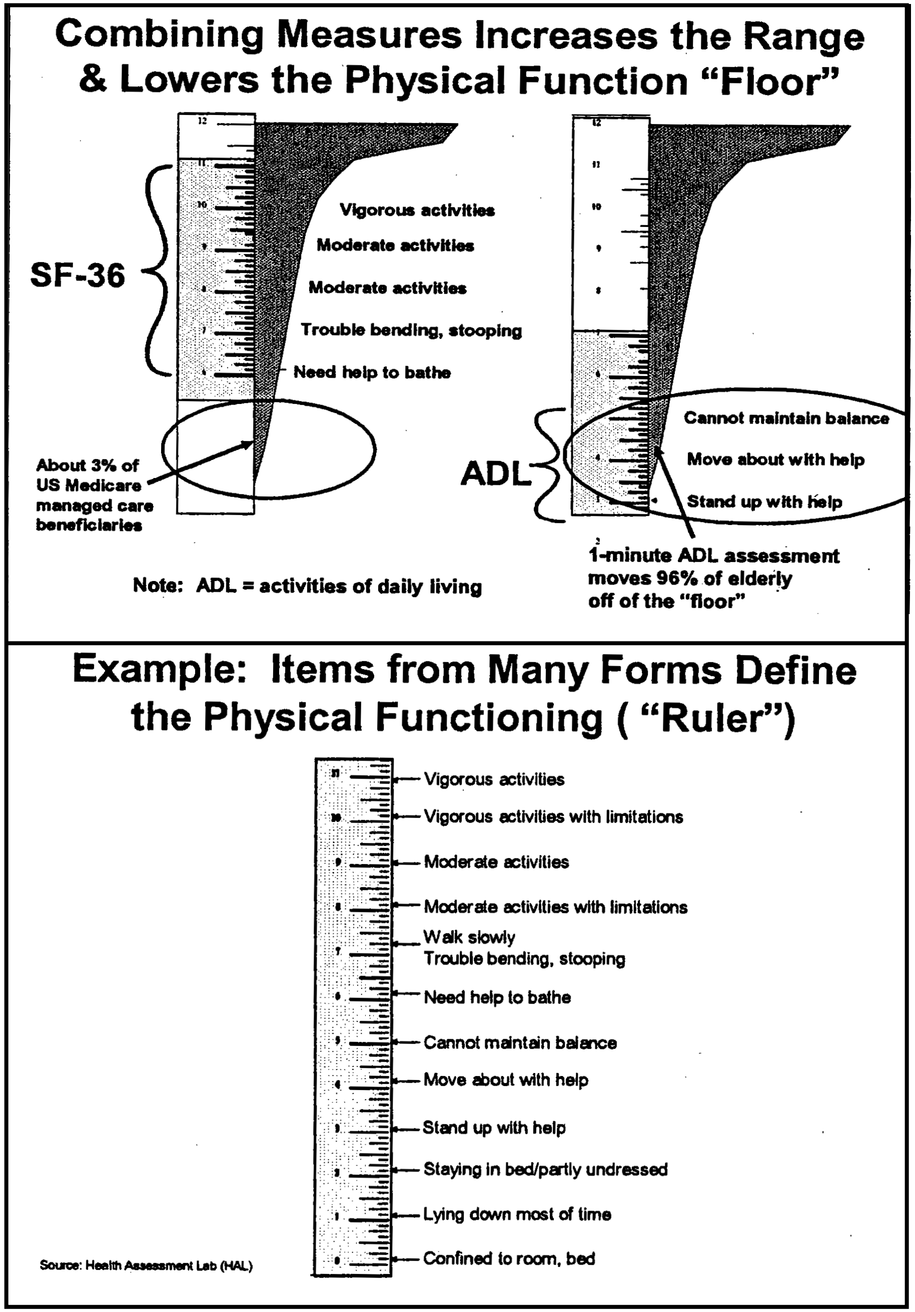
- For fixed-item tests, more items is generally better, but one can get the same precision of measurement with fewer items by using **adaptive tests with items of targeted levels of difficulty**. Different forms across person are given to maximize efficiency.



IRT (and CFA) Help Measure Change AND Maintain Sensitivity across Samples

- **Theta is scaled and interpreted relative to the items**, not relative to the other persons in the sample (is item difficulty at *prob* = .50)
  - This means that given a set of pre-calibrated "linking" items, you can administer different forms of a test and still get comparable Thetas
  - "**Linking items**" → common set of items across forms
  - Although this property is helpful when dealing with 'accidental' alternative forms (e.g., changed response options, dropped items), Linking items can be used advantageously as well
  - Here, **we 'grow' a test over time** within a sample:





Combining Measures Increases the Range & Lowers the Physical Function "Floor"

Note: ADL = activities of daily living

Example: Items from Many Forms Define the Physical Functioning ("Ruler")

Source: Health Assessment Lab (HAL)

Linking Thetas across Tests

SF-36: measure of *higher* physical functioning

ADL: measure of *lower* physical functioning

Don't choose: Administer a core set of linking items from both tests to a single sample

Linking items then form a common metric

- More precision than single test
- Allows for comparisons across groups or studies

See Mungas & Reed (2000) for an example of linking over forms

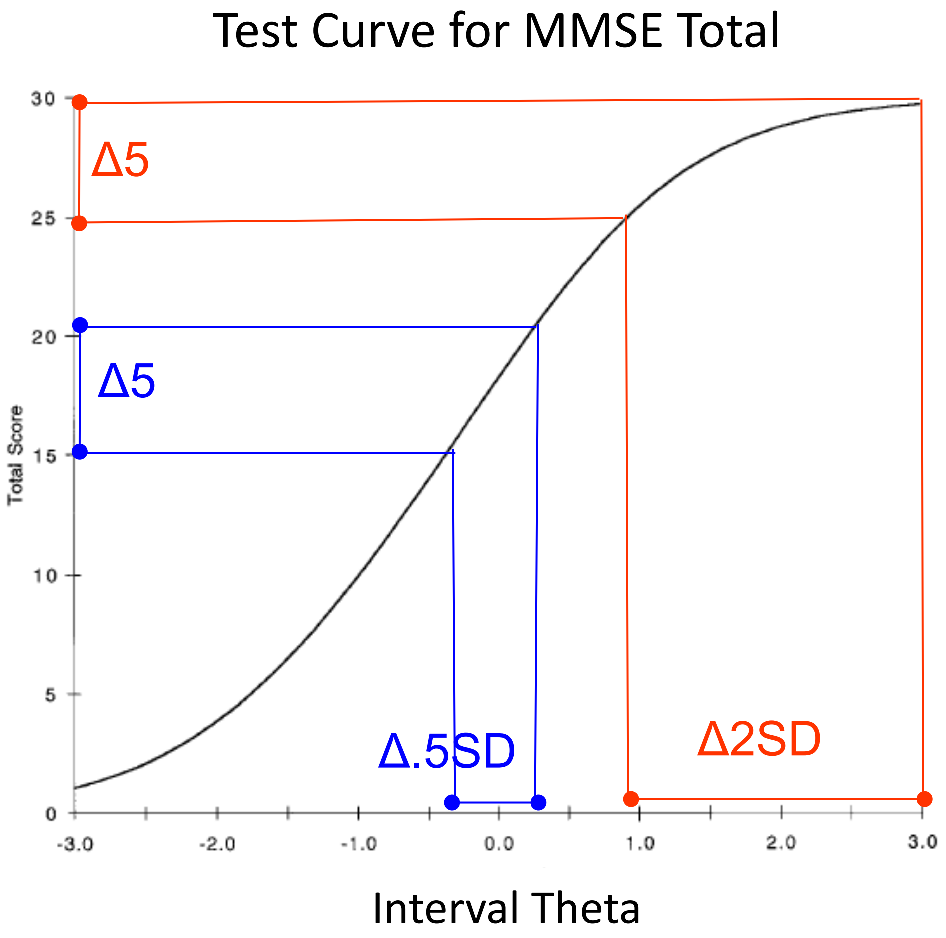
Another Benefit of IRT (and CFA)

- **IRT**: If the model fits, the scale of **Theta is linear/interval**
  - Supports mathematical operations that assume interval measurement
  - Same ordering of persons as in raw scores, but the distances between persons are likely to be different, especially at the ends

- **CTT**: **Sum scores** have an **ordinal** relationship to the latent trait at best
  - Does not support operations that assume interval measurement, which can bias tests of mean differences, regression slopes, etc.
  - Spurious interactions can result in tests of mean differences if groups differ in how well they are measured (i.e., floor and ceiling effects)

- Bottom line: Measurement matters for testing everyday hypotheses, NOT just when fitting measurement models for specific issues

Example from Mungas & Reed (2000)

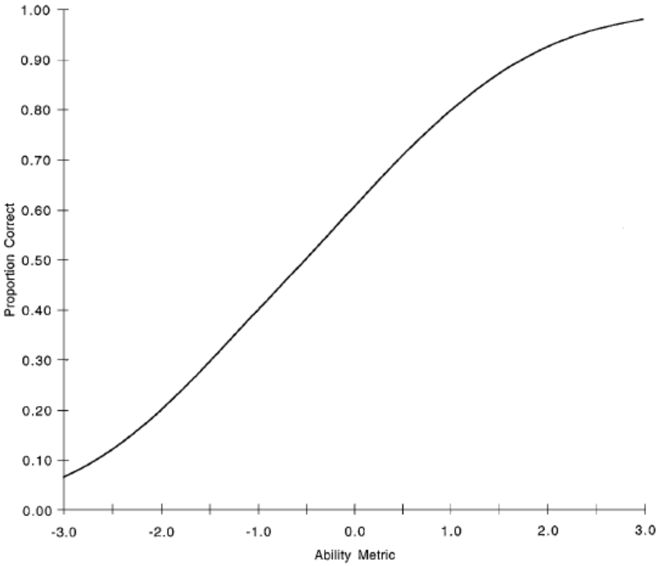


The bottom and top of the MMSE total score (ordinal) are "squished" relative to the latent trait scale (interval).

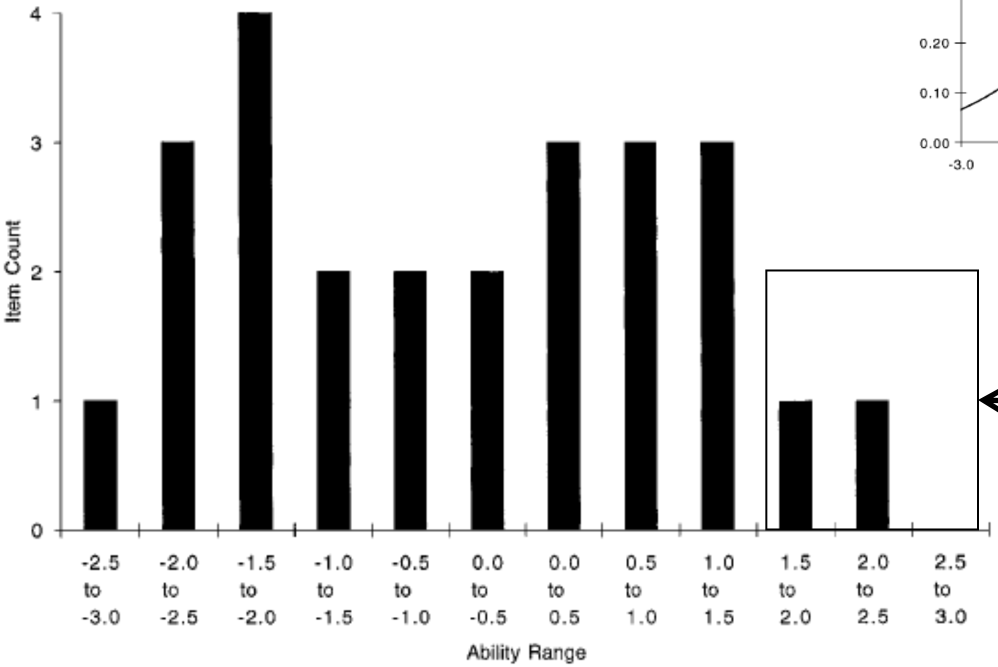
This means that one-unit changes along the MMSE total do not really have the same meaning across the latent trait, which makes many kinds of comparisons problematic.

Example from Mungas & Reed (2000)

Right: They combined 3 tests to get better measurement, as shown in the test curve →



Below: Items at each trait location contribute to scale's capacity to differentiate persons in ability at each point in the continuum.



There is a hole near the top, which explains the flattening of the curve (less information there).

Relating Item Factor Analysis (IFA) to Item Response Models (IRT)

- CFA is to linear regression as IRT is to logistic regression, right?

- **Linear regression model** and **CFA model** (for continuous responses):

$$y_{is} = \beta_{0i} + \beta_{1i}X_s + e_{is} \qquad y_{is} = \mu_i + \lambda_i F_s + e_{is}$$

- **Logistic regression model** (for 0/1 responses, so there is no e_{is} term):

$$\text{Logit}(y_{is} = 1) = \beta_{0i} + \beta_{1i}X_s$$

- **2-PL IRT model** (for 0/1 responses, so there is no e_{is} term):

$$\text{Logit}(y_{is} = 1) = a_i(\theta_s - b_i)$$

Why does this IRT model look so different than the CFA model? Here's how these all relate…

Relating IFA and IRT

- **Linear regression model** and **CFA model**:

  $y_{is} = \beta_{0i} + \beta_{1i}X_s + e_{is}$ $\quad\quad$ $y_{is} = \mu_i + \lambda_i F_s + e_{is}$

- **Binary regression models** and **Our new IFA models:**

  $\text{Logit}(y_{is} = 1) = \beta_{0i} + \beta_{1i}X_s$ $\quad\quad$ $\text{Logit}(y_{is} = 1) = -\tau_i + \lambda_i F_s$

  $\text{Probit}(y_{is} = 1) = \beta_{0i} + \beta_{1i}X_s$ $\quad\quad$ $\text{Probit}(y_{is} = 1) = -\tau_i + \lambda_i F_s$

- **IRT models:**

  2PL: $\text{Logit}(y_{is} = 1) = a_i(\theta_s - b_i)$

  Ogive: $\text{Probit}(y_{is} = 1) = a_i(\theta_s - b_i)$

Logit to Probability:

$$\text{prob} = \frac{\exp(\text{logit})}{1 + \exp(\text{logit})}$$

- In CFA, item loading $\boldsymbol{\lambda_i}$ = "**discrimination**" and item intercept $\boldsymbol{\mu_i}$ = "**difficulty**", but difficulty was backwards (easier or less severe items had higher means)…
- In IFA for binary items within Mplus, the **intercept** $\boldsymbol{\mu_i}$ (which was really "**easiness**") becomes a "**threshold**" $\boldsymbol{\tau_i}$ that really IS "**difficulty**": $\boldsymbol{\mu_i = -\tau_i}$

  → this provides continuity of direction with the IRT difficulty values
- The IRT and IFA models get re-arranged into each other as follows…

From IFA to IRT

IFA with "easiness" **intercept μ_i**: $\textbf{Logit or Probit } y_{is} = \mu_i + \lambda_i F_s \quad \mu_i = -\tau_i$

IFA with "difficulty" **threshold τ_i**: $\textbf{Logit or Probit } y_{is} = -\tau_i + \lambda_i F_s$

IFA model with "difficulty" thresholds can be written as a **2-PL IRT Model:**

IRT model: **IFA model:**

$$\textbf{Logit or Probit } y_{is} = a_i(\theta_s - b_i) \quad = \; -\underbrace{a_i b_i}_{\tau_i} + \underbrace{a_i}_{\lambda_i}\theta_s$$

a_i = discrimination
b_i = difficulty
θ_s = F_s latent trait

Convert IFA to IRT: **Convert IRT to IFA:**

$$a_i = \lambda_i * \sqrt{\text{Theta Variance}} \qquad \lambda_i = \frac{a_i}{\sqrt{\text{Theta Variance}}}$$

$$b_i = \frac{\tau_i - (\lambda_i * \text{Theta Mean})}{\lambda_i * \sqrt{\text{Theta Variance}}} \qquad \tau_i = a_i b_i + \frac{a * \text{Theta Mean}}{\sqrt{\text{Theta Variance}}}$$

Note: These formulas rescale a_i and b_i so that theta M=0, VAR=1.

If you don't want to rescale theta, use M=0 and VAR=1 to keep your current scale.

Thus, IFA = IRT

IRT: **IFA:**

$$\textbf{Logit or Probit } \mathbf{y_{is}} = \mathbf{a_i(\theta_s - b_i)} = -\underbrace{\mathbf{a_i b_i}}_{\boldsymbol{\tau_i}} + \underbrace{\mathbf{a_i}}_{\boldsymbol{\lambda_i}}\boldsymbol{\theta_s}$$

- An item factor model for binary outcomes is the same as a two-parameter IRT model, so you can keep both camps happy:
  - IFA loadings $\boldsymbol{\lambda_i}$ → 2-PL IRT discriminations $\mathbf{a_i}$
  - IFA thresholds $\boldsymbol{\tau_i} = -\boldsymbol{\mu_i}$ → 2-PL IRT difficulties $\mathbf{b_i}$

- CFA/SEM crowd? Call it $\textbf{Logit or Probit } \mathbf{y_{is}} = -\boldsymbol{\tau_i} + \boldsymbol{\lambda_i}\mathbf{F_s}$
  - "I did IFA" → Report item "factor loadings" $\boldsymbol{\lambda_i}$ and "thresholds" $\boldsymbol{\tau_i}$
- IRT crowd? Call it $\textbf{Logit or Probit } \mathbf{y_{is}} = \mathbf{a_i(\theta_s - b_i)}$
  - "I did IRT" → Report item "discriminations" $\mathbf{a_i}$ and "difficulties" $\mathbf{b_i}$

3 Kinds of Output in Mplus and Lavaan

- **IFA unstandardized solution:**
  - **Item threshold $\boldsymbol{\tau_i}$** = expected logit or probit of **y=0** when Theta=0
  - **Item loading $\boldsymbol{\lambda_i}$** = Δ in logit or probit of y_{is}=1 for a 1-unit Δ in Theta
  - Item residual variance is not estimated, but is 3.29 in logit or 1.00 in probit for y*

- **IFA standardized solution:**
  - Variance of logit or probit (y_{is}=1) → (λ_i^2 * Theta Variance) + (3.29 or 1)
  - **std. $\boldsymbol{\tau_i}$** = unstd. $\boldsymbol{\lambda_i}$ / SD(Logit or Probit Y) → not usually interpreted
  - **std. $\boldsymbol{\lambda_i}$** = unstd. $\boldsymbol{\lambda_i}$ * SD(Theta) / SD(Logit or Probit Y)

    → correlation of logit/probit of item response with Theta

The IFA solution **cannot** be used to compute Omega.

- **IRT solution** (only one type; only given for binary items):
  - $\mathbf{b_i}$ = Theta at which prob(y_{is}=1) = .50 or logit or probit = 0
  - $\mathbf{a_i}$ = Δ in logit or probit of y=1 for a 1-unit Δ in Theta

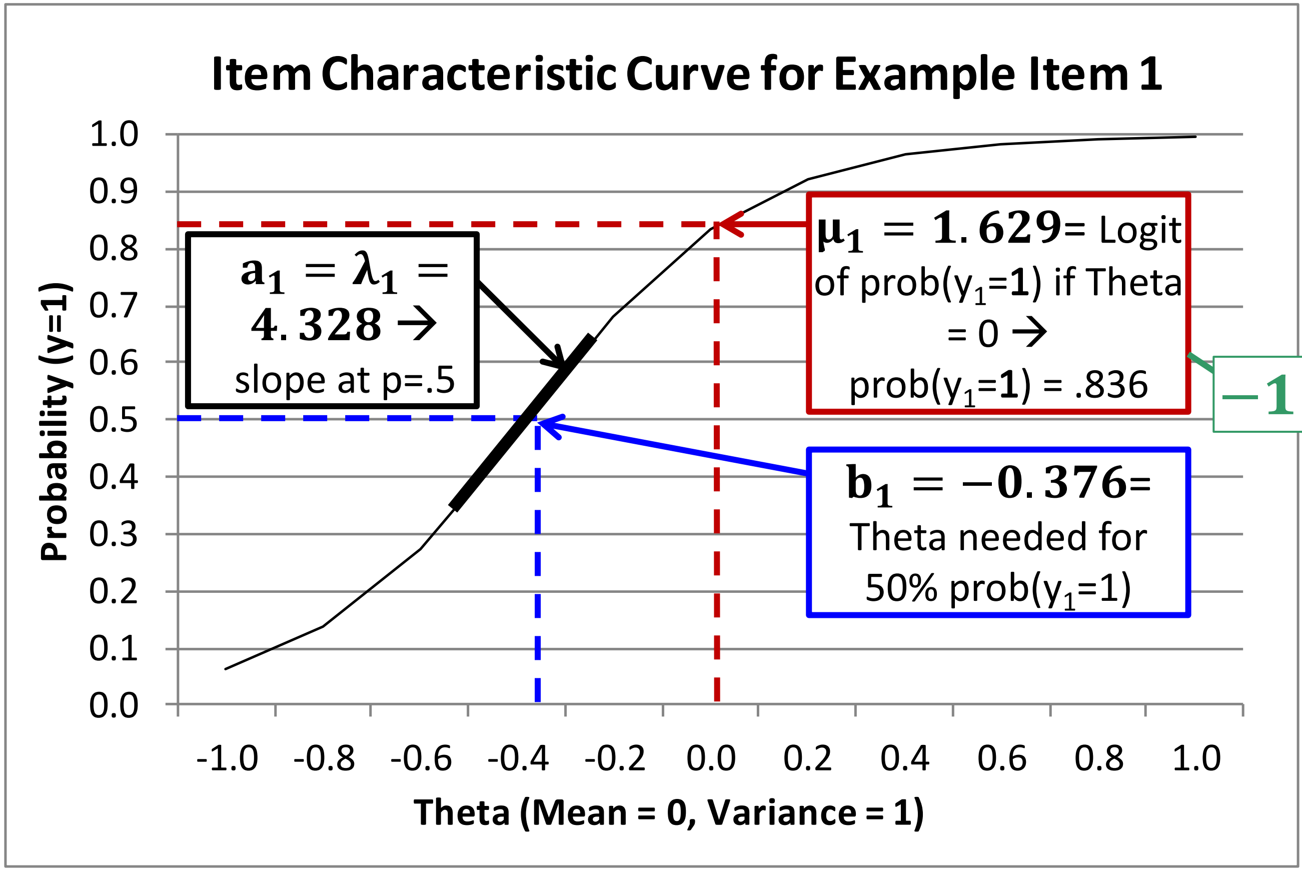
    = slope of item characteristic curve at $\mathbf{b_i}$ location

Item Parameter Interpretations

IFA model with **loading** and "easiness" **intercept μ_i**: $\textbf{Logit } \mathbf{y_{is}} = \boldsymbol{\mu_i} + \boldsymbol{\lambda_i}\mathbf{F_s}$

IFA model with **loading** and "difficulty" **threshold τ_i**: $\textbf{Logit } \mathbf{y_{is}} = -\boldsymbol{\tau_i} + \boldsymbol{\lambda_i}\mathbf{F_s}$

2-PL IRT model with **discrimination** and **difficulty**: $\textbf{Logit } \mathbf{y_{is}} = \mathbf{a_i}(\boldsymbol{\theta_s} - \mathbf{b_i})$



$\mathbf{a_1 = \lambda_1 = 4.328} \rightarrow$ slope at p=.5

$\boldsymbol{\mu_1} = \mathbf{1.629}$= Logit of prob($y_1$=**1**) if Theta = 0 → prob(y_1=**1**) = .836

$\mathbf{b_1 = -0.376}$= Theta needed for 50% prob(y_1=1)

$\mathbf{-1} *$

$\boldsymbol{\tau_1} = \mathbf{-1.629}$= Logit of prob($y_1$=**0**) if Theta = 0 → prob(y_1=**0**) = .164

From IRT to IFA:

$\lambda_i = a_i$

$\tau_i = a_i b_i$

Item Parameter Interpretation

IFA model with **loading** and "easiness" **intercept $\boldsymbol{\mu_i}$**: $\mathbf{Logit\ y_{is} = \ \mu_i + \lambda_i F_s}$

IFA model with **loading** and "difficulty" **threshold $\boldsymbol{\tau_i}$**: $\mathbf{Logit\ y_{is} = -\tau_i + \lambda_i F_s}$

2-PL IRT model with **discrimination** and **difficulty**: $\mathbf{Logit\ y_{is} = a_i(\theta_s - b_i)}$

- IFA and IRT item slope parameters are interpreted similarly:
  - IFA loading λ_i= Δ in logit or probit of y_{is}=1 for a 1-unit Δ in Theta
  - IRT discrimination a_i = slope of ICC at prob=.50 (logit or probit = 0)
- IFA and IRT item location parameters are interpreted differently:
  - **IFA intercept $\boldsymbol{\mu_i}$**= logit or probit of **y_{is}=1** when **Theta = 0**
  - **IFA threshold $\boldsymbol{\tau_i}$**= logit or probit of **y_{is}=0** when **Theta = 0**
  - **IRT difficulty $\mathbf{b_i}$ = amount of Theta needed** for logit or probit of y_{is}=1
    - So $\mathbf{b_i}$ difficulty values are more interpretable as measures of **location**

CFA vs. IRT/IFA vs. ???

- CFA assumes continuous, normally distributed item responses
  - Robust ML can be used to adjust fit statistics and parameter SEs for non-normality, but it's still a **linear model** for the Factor predicting Y
  - A linear model may not be plausible for Likert item responses (i.e., the model-predicted responses may extend beyond the possible response options for possible Factor levels)

- IRT/IFA assumes categorical, binomial/multinomial item responses
  - **Linear model between Theta and logit/probit(y) instead**
  - Because Likert item responses are bounded and only ordinal, not interval, IRT/IFA should probably be used for this kind of data
  - CFA may not be too far off given ≥ 5 normally distributed responses, but then you can't see how useful your answer choices are (stay tuned)

- For non-normal but continuous (not categorical) responses, other latent trait measurement models are available (stay tuned)

Summary: Binary IRT/IFA Models

- IRT/IFA are a family of models that specify the relationship between the latent trait (“Theta”) and a link-transformation of probability of Y
  - **Linear** relationship between Theta and **Logit or Probit** (Y=1)
    → **nonlinear** relationship between Theta and **Probability** (Y=1)

- The form of the relationship depends on:
  - At least the location on the latent trait (given by b_i or τ_i)
  - Perhaps the strength of relationship may vary across items (given by a_i or λ_i)
    - If not, its a “1-PL” or “Rasch model” → assumes tau-equivalence
  - Also maybe lower and upper asymptotes (c_i and d_i) → but hope not!

- Because the slopes are non-linear, this implies that **reliability** (now called “test information”) **must vary** across levels of theta
  - So items are not just “good” or “bad”, but are “good” or “bad” for whom?

- **Now what about model fit??? Let’s talk estimation first…**

What all do we have to estimate?

- For example, a 7-item binary test and a 2-PL model, (assuming we fix the Theta distribution to mean=0 and variance=1):
  - 7 item discriminations (a_i) and 7 item difficulties (b_i) = 14 parameters

- **Item parameters** are **FIXED effects** → specific item inference
  - Missing data can lead to different numbers of total items across persons

- What about the all the individual person **Thetas**?
  - The individual factor scores are <u>not</u> part of the model—in other words, Theta scores are modeled as **RANDOM effects** (= U's in MLM)
  - Thus, our inference is about the distribution of the latent traits in the population of persons, which we assume to be multivariate normal
  - i.e., we care about the **Theta means, variances, and covariances** in the sample, but **not** about the Theta scores for each **individual** per se

Estimation: Items, then People

3 full-information item estimation methods:

- "**Full-information**" → uses individual item responses
- 3 methods differ with respect to how they handle unknown thetas
- First, two less-used and older methods:
  - **"Conditional" ML →** *Theta? We don't need no stinking theta…*
    - Uses total score as "Theta" (so can't include people with all 0's or all 1's)
    - Thus, is only possible within Rasch models (where total is sufficient for theta)
    - If Rasch model holds, estimators are consistent and efficient and can be treated like true likelihood values (i.e., can be used in model comparisons)
  - **"Joint" ML →** *Um, can we just pretend the thetas are fixed effects?*
    - Iterates back and forth between persons and items (each as fixed effects) until item parameters don't change much—then calls it done (i.e., converged)
    - Many disadvantages: estimators are biased, inconsistent, with too small SEs and likelihoods that can't be used in model comparisons
    - More persons → more parameters to estimate, too → so bad gets even worse

Marginal ML Estimation (with Numeric Integration)

- Gold standard of estimation (and used in Mplus and SAS NLMIXED)
  - This is the same idea of multivariate height, just using a different distribution than multivariate normal for the log-likelihood function

- Relies on two assumptions of **independence**:
  - Item responses are independent after controlling for Theta: “local”
    - This means that the joint probability (likelihood) of two item responses is just the probability of each multiplied together
  - Persons are independent (no clustering or nesting)
    - You can add random effects to handle dependence, but then the assumption is “independent after controlling for random effects”

- Doesn’t assume it knows the individual thetas, but it does assume that the theta *distribution* is (multivariate) normal

Marginal ML via Numeric Integration

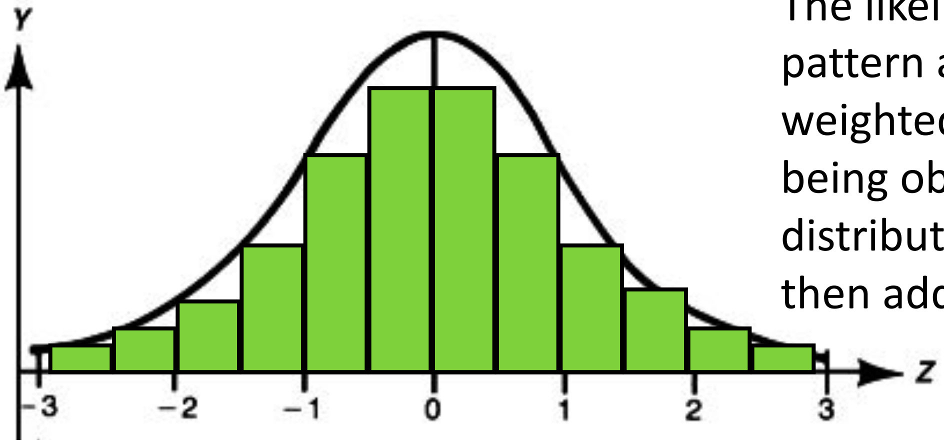
- Step 1: Select starting values for all item parameters (e.g., using CTT values)
- Step 2: Compute the **likelihood for each person** given by the *current* parameter values (using start values or updated values later on)
  - IRT model gives probability of response given item parameters and Theta
  - To get likelihood per person, take each predicted probability and plug them into:
    Likelihood (all responses) = Product over items of: $p^{y}(1-p)^{1-y}$
  - But we don't have Theta yet! No worries: computing the likelihood for each set of possible parameters requires *removing* the individual Thetas from the model equation—by ***integrating*** across the possible Theta values for each person
  - Integration is accomplished by "Gaussian Quadrature" → summing up rectangles that approximate the integral (the area under the curve) for each person
- **Step 3:** Decide if you have the right answers, which occurs when the sum of the log-likelihoods changes very little across iterations (i.e., it converges)
- **Step 4:** If you aren't converged, choose new parameters values
  - Newton-Rhapson or Fisher Scoring (calculus), EM algorithm (Thetas =missing data)

“Marginal” ML Estimation

- More on Step 2: Divide the Theta distribution into rectangles

  → “**Gaussian Quadrature**” (# rectangles = # “**quadrature points**”)

  - Divide the whole distribution into rectangles, and then take the most likely section for each person and rectangle that more specifically
    - This is “**adaptive quadrature**” and is computationally more demanding, but gives more accurate results with fewer rectangles (Mplus uses 15)



The likelihood of each person’s response pattern at each Theta rectangle is then weighted by that rectangle’s probability of being observed (as given by the normal distribution). The weighted likelihoods are then added together across all rectangles.

→ ta da! “**numeric integration**”

  - Unfortunately, each additional Theta or Factor adds another dimension of integration (so 2 factors = 15*15 rectangles to try at each iteration)

Example of Numeric Integration

1. Start values for item parameters (for simplicity, assume a=1):
   - Item 1: mean = .73 → logit = +1, so starting $b_1 = -1$
   - Item 2: mean = .27 → logit = −1, so starting $b_2 = +1$
2. Compute per-person likelihood using item parameters and possible Thetas (−2,0,2) using IRT model: $\text{logit}(y_{is} = 1) = a(\theta - b_i)$

			IF y=1	**IF y=0**	**Likelihood**	**Theta**	**Theta**	**Product**
	Theta = -2	**Logit**	**Prob**	**1-Prob**	**if both y=1**	**prob**	**width**	**per Theta**
Item 1 b = -1	(-2 - -1)	-1	0.27	0.73	0.0127548	0.05	2	0.001275
Item 2 b = +1	(-2 - 1)	-3	0.05	0.95				
	Theta = 0	**Logit**	**Prob**	**1-Prob**				
Item 1 b = -1	(0 - -1)	1	0.73	0.27	0.1966119	0.40	2	0.15729
Item 2 b = +1	(0 - 1)	-1	0.27	0.73				
	Theta = +2	**Logit**	**Prob**	**1-Prob**				
Item 1 b = -1	(2 - -1)	3	0.95	0.05	0.6963875	0.05	2	0.069639
Item 2 b = +1	(2 - 1)	1	0.73	0.27				
Overall Likelihood (Sum of Products over All Thetas):								**0.228204**

(then multiply over all people)

(repeat with new values of item parameters until find highest overall likelihood)

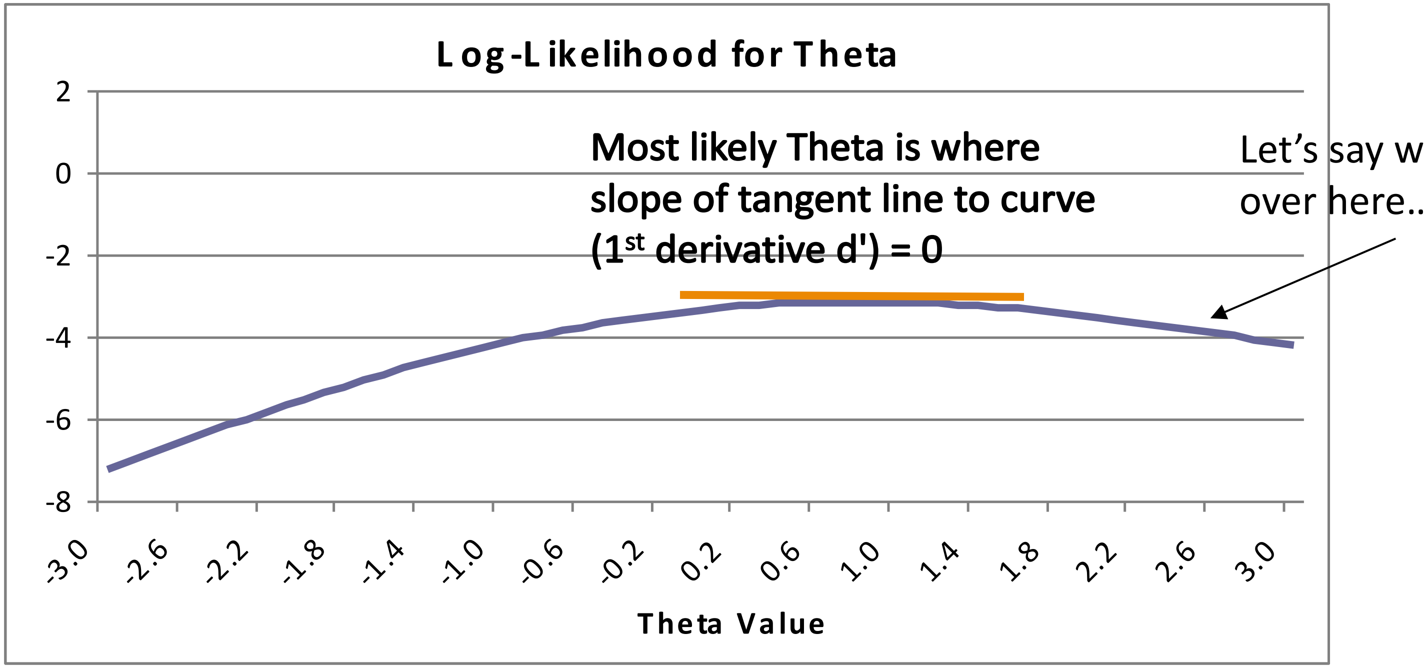
Once we have the items parameters, we need some Thetas…

- Let’s say we are searching for Theta given observed responses to 5 items with known difficulty values, so we try out two possible Thetas
  - **Step 1**: Compute prob(Y) using IRT model given each possible Theta
    - b_1 = -2, θ = -1: Logit(Y=1) = (-1 − -2) = 1, so p = .73
    - b_5 = 2, θ = -1: Logit(Y=1) = (-1 − 2) = -3, so p = .05 → 1-p = .95 (for Y=0)
  - **Step 2**: Multiple item probabilities together → product = “likelihood”
    - Products get small fast, so can take the log, then add them instead
  - **Step 3**: See which Theta has the highest likelihood (here, +2)
    - More quadrature points → better estimate of Theta
  - **Step 4**: Because people are independent, we can multiply all their response likelihoods together and solve all at once

Item	b	Y	Term	Value if…	
				θ = -1	θ = +2
1	-2	1	p	0.73	0.98
2	-1	1	p	0.50	0.95
3	0	1	p	0.27	0.88
4	1	1	p	0.12	0.73
5	2	0	1-p	0.95	0.50
Product of values:				0.01	0.30

Theta Estimation via Newton Raphson

- We could calculate the likelihood over wide range of Thetas for each person and plot those likelihood values to see where the peak is…
  - But we have lives to lead, so we can solve it mathematically instead by finding where the slope of the likelihood function (the 1st derivative, d') = 0 (its peak)
- Step 1: Start with a guess of Theta, **calculate 1st derivative d'** at that point
  - Are we there (d' = 0) yet? Positive d' = too low, negative d' = too high

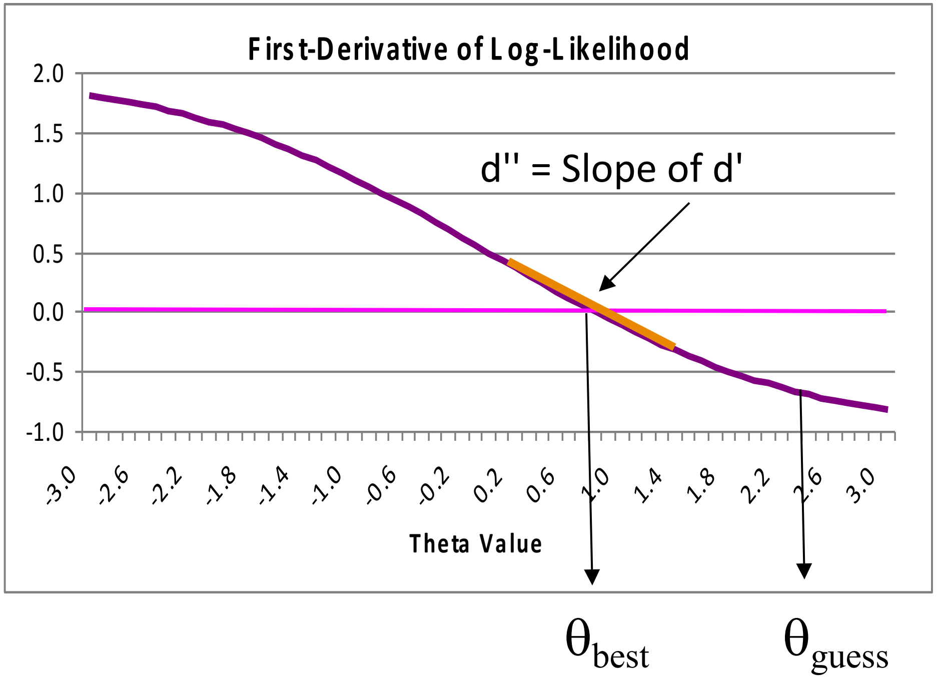


Most likely Theta is where slope of tangent line to curve (1st derivative d') = 0

Let’s say we started over here…

Theta Estimation via Newton Raphson

- Step 2: **Calculate the 2nd derivative** (slope of slope, d'') at that point
  - Tells us **how far off we are**, and is used to figure out how much to adjust by
  - d'' will always be negative as approach top, but d' can be positive or negative
- Calculate new guess of Theta: $\theta_{new} = \theta_{old} - (d'/d'')$
  - If (d'/d'') < 0 →Theta increases
    If (d'/d'') > 0 →Theta decreases
    If (d'/d'') = 0 then you are done
- **2nd derivative d'' also tells you how *good* of a peak you have**
  - Need to know where your best Theta is (at d'=0), as well as how precise it is (from d'')
  - If the function is flat, d'' will be smallish
  - **Want large d'' because 1/SQRT(d'') = Theta's SE**



Theta Estimation: ML with Help

- ML is used to come up with most likely Theta given observed response pattern and item parameters…

  …but can't estimate Theta if answers are all 0's or all 1's

- **Prior distributions** to the rescue!
  - Multiply likelihood function for Theta with prior distribution (usually we assume normal)
  - Contribution of the prior is minimized with increasing items, but allows us to get Thetas for all 0 or all 1 response patterns

- Note the implication of this for what Theta really is for each person:
  - **THETA IS A DISTRIBUTION, NOT A VALUE!**
  - Although we can find the most likely value, we can't ignore its probabilistic nature or how good of an estimate it is (how peaked)
    - SE is constant for CFA factor scores, but SE is NOT constant for IRT Thetas
  - **THIS IS WHY YOU SHOULD AVOID OUTPUTTING THETAS (PSOT!)**

Theta Estimation: 3 Methods

- **ML**: Maximum Likelihood Scoring
  - Uses just item parameters to come up with Thetas
  - Can't estimate Theta if none or all are answered correctly

- **MAP**: Maximum a Posteriori Scoring
  - Combine ML estimate with a continuous normal prior distribution
  - Theta estimate is mode of combined posterior distribution
  - Theta will be regressed toward mean if reliability is low
  - Is used in Mplus WLSMV

- **EAP**: Expected A Posteriori Scoring
  - Combine ML estimate with a 'rectangled' normal prior distribution
  - Theta estimate is mean of combined posterior distribution
  - Is used in Mplus ML for CFA or IRT/IFA (and is best version)

Model Comparisons in IRT: Relative Model Fit via −2ΔLL Tests

- **Nested models** can be compared with the same −2ΔLL tests we used in CFA → without the "robust" part of ML, so they get simpler (scale factor=1)
  - Step 1: Calculate $-2\Delta LL = -2(LL_{fewer} - LL_{more})$
  - Step 2: Calculate $\Delta df = df_{more} - df_{fewer}$ (given as "# free parms")
  - Compare −2ΔLL with df = Δdf to χ^2 critical values (or excel CHIDIST)
  - Add 1 parameter? −2ΔLL(1) > 3.84, add 2: −2ΔLL(2) > 5.99…

- If **adding** a parameter, model fit can be **better** or **not better**
- If **removing** a parameter, model fit can be **worse** or **not worse**
- AIC and BIC values (computed from −2LL) can be used to compare non-nested models (given same sample), smaller is better
- No trustable absolute global fit measures available via full information ML for IRT → categorical data can't be summarized via a covariance matrix

Local Model Fit Using ML IRT

- IRT programs (but not Mplus) provide “item fit” and “person fit” statistics
  - Item fit: Predicted vs. observed ICCs—how well do they match?
    Or via inferential tests (Bock Chi-Square Index or BILOG version)
  - Person fit “Z” based on predicted vs. observed response patterns
  - Many require the use of outputted thetas, which makes then problematic

- **Using ML in Mplus**: Local item fit available with **TECH10** output
  - **Univariate item fits**: How well did the model reproduce the observed response proportions? (Not likely to have problems here)
  - **Bivariate item fits**: Contingency tables for pairs of responses → Get χ^2 value for each pair of items for their remaining dependency after controlling for Theta(s)

- Bivariate item fit is the basis of the newest absolute fit statistics discussed by Maydeu-Olivares (2105): M_2 (analogous to χ^2 test), RMSEA_2, and SRMR_2
  - Not currently provided in Mplus; not currently standard practice

What Goes Wrong for Absolute (Global) Model Fit using ML…

- **ML is a full-information estimator, and it is now trying to reproduce the observed item response pattern, <u>not a covariance matrix</u>!**
- Model DF is based on FULL response pattern:
  - DF = # possible observed patterns – # parameters – 1
  - So, for an example of 24 binary items in a 1-PL Model:
    - $\text{Max DF} = 224 - \#\mathrm{a_i} - \#\mathrm{b_i} - 1 = 16{,}777{,}216 - 1 - 24 - 1 = \mathbf{16{,}777{,}190}$!
    - If some cells aren't observed (Mplus deletes them from the χ^2 calculation), then DF may be < Max DF, and thus χ^2 won't have the right distribution
- Pearson χ^2 based on classic formula: (observed – expected)^2 / expected
  - Good luck finding enough people to fill up all possible patterns!
  - Other χ^2 given in output is "Likelihood Ratio" χ^2 , calculated differently
  - Linda Muthén suggests "if these don't match, they should not be used"
  - **χ^2 generally won't work well for assessing absolute global fit in IRT**

Summary: ML for IRT Models

- Full-information Marginal ML with numeric integration for IRT models tries to find the item parameters that are most likely *given the observed item response pattern* → IFA or IRT parameters on logit or probit scales

- Because of the integration (rectangling Theta) required at each step of estimation, it will not be feasible to use ML for IRT models in small samples or for many factors at once (too many rectangles simultaneously)

- IRT using ML does not have agreed-upon measures of absolute global fit
  - Outcomes cannot be summarized by a covariance matrix anymore, only by the possible response patterns instead
  - Usually not enough people to fill up all possible response patterns, so there’s no valid basis for an absolute fit comparison
  - Nested models (on same items) can still have relative fit compared via −2ΔLL

- There is another game in town for IRT in Mplus, however…

Another Alternative: WLSMV

- **WLSMV:** “Weighted Least Square parameter estimates use a diagonal weight matrix and a Mean- and Variance-adjusted χ^2 test”
  - Called “diagonally-weighted least squares” by non-Mplus people

- Translation: **WLSMV** is a **limited-information** estimator that uses a different summary of responses instead → **a “linked” covariance matrix**

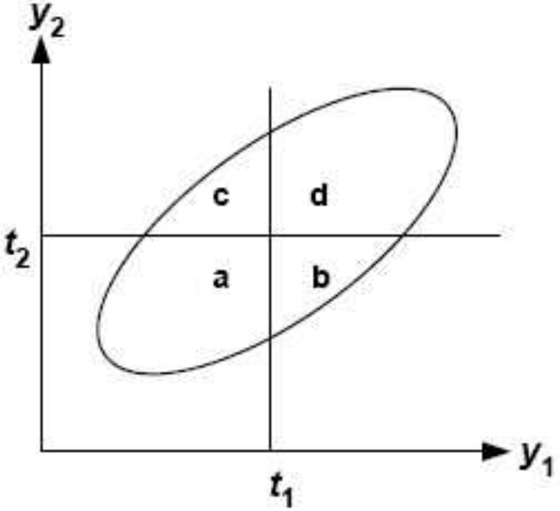
- Fit can then be assessed in regular CFA ways, because what is trying to be reproduced is again a type of covariance matrix
  - Instead of the *full item response pattern* (as in ML)
  - We can then get the typical measures of absolute fit as in CFA

- Normally CFA uses the *observed* covariance matrix of the items…
  - But correlations among binary items will be less than 1 any time p differs between items, so the covariances will be restricted as well…
  - What if we could fit a covariance matrix on the logit or probits instead???

WLSMV Estimation

Data	y_2 = 0	y_2 = 1
y_1 = 0	a	c
y_1 = 1	b	d

Use the observed proportions as the area under the curve of each section of the bivariate distribution to determine what the correlation would be →



- WLSMV first estimates correlation matrix for **probit** of item responses (no logits here)
  - For binary responses → "tetrachoric correlation matrix"
  - For ordinal (polytomous) responses → "polychoric correlation matrix"
- The model then tries to find item parameters to predict this new correlation matrix
- The diagonal W "weight" part then tries to emphasize reproducing latent variable correlations that are relatively well-determined more than those that aren't
  - The full weight matrix is of order z*z, where z is number of elements to estimate
  - The "diagonal" part means it only uses the *preciseness of the estimates themselves*, not the covariances among the "preciseness-es" (much easier, and not a whole lot of info lost)
- The "MV" corrects the χ^2 test for bias arising from this weighting process

More about WLSMV Estimation

- Works much faster than ML when you have small samples or many factors to estimate (because no rectangling is required)
- Does assume missing data are **missing completely at random,** whereas ML assumes only *missing at random* (conditionally random)
- Because a covariance matrix of the probits is used as the input data, we get absolute fit indices as in CFA
  - People tend not to be as strict with cut-off values, though
  - One new one: WRMR is "experimental", but should be < 1 or so

- Model coefficients will be on the **probit scale** instead of logit scale
- Two different model variants are available via the **PARAMETERIZATION IS** option on the **ANALYSIS** command
  - "**Delta**" (default): variance (Y*) = factor + error = 1 = "marginal parameterization"
  - "**Theta**": error variance = 1 instead = "conditional parameterization"
    - **WE WILL USE THIS ONE TO HELP SIMPLIFY IRT CONVERSIONS**

Model Comparisons with WLSMV using DIFFTEST in Mplus

- Not the same process! DF is NOT calculated in usual way, and model fit is not compared in the usual way
  - Absolute χ^2 model fit values are meaningless—they are not comparable!
  - Difference in model χ^2 are not distributed as χ^2

- Here's how you do nested model comparisons in WLSMV:
  - Step 1: Estimate model with *more* parameters, adding this command:
    - SAVEDATA: DIFFTEST=more.dat; → Saves needed derivatives
  - Step 2: Estimate model with *fewer* parameters, adding this command:
    - ANALYSIS: DIFFTEST=fewer.dat; → Uses those derivatives to do $\Delta\chi^2$ test
  - Step 2 model output will have a new χ^2 difference test in it that you can use, with df difference to compare to a χ^2 distribution

Model Comparisons with WLSMV using DIFFTEST in lavaan

- All we need to use is the anova() function

- Fin.

Assessing Local Model Fit

- **The need to check local model fit is the same in IRT/IFA as in CFA**

- **Using ML**: Local item fit in Mplus available with **TECH10** option
  - **Univariate item fits**: How well did the model reproduce the observed response frequencies? (Not likely to have problems here if each item has own location)
  - **Bivariate item fits**: Contingency tables for pairs of responses → Get χ^2 value for each pair of items for their remaining dependency after controlling for Theta(s)

- **Under WLSMV**: Residual correlation matrix via the RESIDUAL option on OUTPUT statement (just as in CFA)
  - Predicted and residual (left-over) item correlations given in *correlation* metric
  - Look for large residual correlations in absolute value (but no significance tests)
  - Will be MUCH easier to do for many items than bivariate fit in ML

Residual Covariances in IRT/IFA

- Additional relationships between items can be included:
  - Via **residual covariances** (the same as in CFA) when using **WLSMV** because the model is being estimated on the tetrachoric/polychoric correlation matrix (so the residuals of the underlying probit can covary, even if item residual variances are not being estimated)
  - Residual covariances are not allowed when using maximum likelihood
  - Instead, you can specify "**method factors**" (in WLSMV or ML), also known as a "bifactor model"
  - In lavaan we use the ~~ like in CFA
- Here is an example using WLSMV to demonstrate both ways:

```
! Primary factor/theta
Trait BY item1-item5*;
[Trait@0]; Trait@1;
! Residual covariance
item2 WITH item3*;
```

```
! Primary factor/theta
Trait BY item1-item5*;
[Trait@0]; Trait@1;
! Uncorrelated factor to
  create residual covariance
ResFact BY item2@1 item3@1;
[ResFact@0]; ResFact*;
ResFact WITH Trait@0;
```

Residual Covariances in IRT/IFA

```
! Primary factor/theta
Trait BY item1-item5*;
[Trait@0]; Trait@1;
! Uncorrelated factor to
  create residual covariance
ResFact BY item2@1 item3@1;
[ResFact@0]; ResFact*;
ResFact WITH Trait@0;
```

For models with many method factors, add the **ANALYSIS:** option **MODEL=NOCOVARIANCES** to made all factors **un**correlated by default (instead of correlated by default as usual)

```
TRAIT    BY
    ITEM1      0.994      0.078     12.724      0.000
    ITEM2      2.138      0.148     14.459      0.000
    ITEM3      1.823      0.125     14.527      0.000
    ITEM4      1.106      0.090     12.311      0.000
    ITEM5      0.232      0.045      5.200      0.000

RESFACT  BY
    ITEM2      1.000      0.000    999.000    999.000
    ITEM3      1.000      0.000    999.000    999.000

RESFACT  WITH
    TRAIT      0.000      0.000    999.000    999.000

Variances
    TRAIT      1.000      0.000    999.000    999.000
    RESFACT    1.996      0.314      6.357      0.000
```

To create a negative residual covariance, fix the ResFact loadings to 1 and −1 instead.

The variance of ResFact is the positive residual covariance between items 2 and 3.

IRT/IFA Model Estimation: Summary

- Full-information Marginal ML estimation with numeric integration provides:
  - '**Best guess**' at to the value of each item and person parameter
  - **SE** that conveys the uncertainty of that prediction

- The '**best guesses**' for the model parameters do not depend on the sample:
  - Item estimates do not depend on the particular individuals that took the test
  - Person estimates do not depend on the particular items that were administered
  - Thus, model parameter estimates are sample-invariant

- The **SEs** for those model parameters DO depend on the sample
  - Item parameters will be estimated less precisely where there are fewer individuals
  - Person parameters will be estimated less precisely where there are fewer items

- **WLSMV** in Mplus is a limited-estimation approach for IFA or IRT models
  - Uses an estimated tetrachoric correlation matrix as input for the factor analysis
  - Works better for many factors than ML (but can be less trustworthy overall)